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OPTIMAL CONTROL OF WATER POLLUTION IN A RIVER SYSTEM

by

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled Optimal Control of Water Pollution in a River System submitted by Ronald L. Lawson in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The optimal dumping of organic waste and the artifical aeration of a river in order to maintain adequate levels of dissolved oxygen in the water are considered in this thesis. The problem is formulated as an optimal control problem with several performance indices. Available results on multi-cost optimization theory are used in the present study. Optimum profiles of BOD (Biochemical oxygen demand) and DO (dissolved oxygen) are obtained for several objective functionals. These functionals are expressions relating the chosen performance indices. In previous work in this area, the approach has been to use a single performance index. However, since a river has to fulfil many requirements,some often contradictory, the multi-cost approach used in this thesis has a decided advantage.

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CHAPTER (1)

INTRODUCTION

1.1 Man and his Environment

It is becoming increasingly evident that the environment is not limitless.* In some areas, pollutants and associated destructive agents are affecting the health, recreational life and the opportunity to work of each and every man. Air is a continuous requirement of man while water is consumed selectively in different forms along with food from the soil during certain times of the day. These basic requirements for life are endangered, since the ecosystem is being altered by man's activities.** Thus engineers are becoming concerned about this problem and are attempting to do their share to correct the wrongs which have resulted from man's previous indifference to the ecosystem.

The soil has been relegated to the role of the recipient of many products resulting from man's activities such as solid refuse disposal from both urban and rural sources, chemical agents used in the production of food, and contamination from particles settling out of the atmosphere. It has been recently discovered that the use of pesticides has caused harmful residues to be concentrated in the food chain. The pollution of air is caused by burning of fuels for transportation, heat and power.

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* Environment: a continually changing complex of all conditions and influences interacting with an organism.

** Ecosystem: a relationship between living things and their nonliving but supporting environment.

Sources of air pollution emit some or all of the following: sulfur dioxide, particle matter, carbon monoxide, photochemical oxidants, hydrocarbons and nitrogen dioxide [2].* It has been speculated that the oxygen - carbon dioxide cycle in nature might be upset due to these pollutants. In the case of water, municipalities and industries use lakes, rivers and oceans for the dilution of many liquid wastes which they generate. By using nearby rivers or lakes for the disposal of liquid wastes industries such as pulp and paper have introduced toxic substances like mercury into the water. This has resulted in the recall of some food products and the banning of others, for example certain types of fish, from the market place.

1.2 Water Pollution

The general quality of water in lakes, rivers and oceans, depends mainly on meteorological factors, municipal uses and industrial uses in their vicinity. The deposition of certain dissolved gases and particles from the atmosphere and the washing of organic matter, suspended solids, and chemical residues from the surface of the earth are the main ingredients of meteorological pollution of surface waters. Municipal discharges contain organic matter, infectious agents, suspended solids and detergents, while industrial discharges are characterized by chemical residues, metal ions, heat, organic and inorganic matters.

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* Numbers in rectangular brackets refer to references listed under Bibliography at the end of this thesis.

We shall discuss one aspect of this problem, namely the addition of organic waste to a river. However before this physical problem is defined, it seems advantageous to describe what is known as the "organic cycle of nature" [26].

Organic Cycle

When food is consumed, the outcome is the liberation of energy and the replacement of worn out cells. Nitrogen, phosphorous and other elements are used to repair the deteriorated cells. The replaced materials are discharged through the human system as waste products. The energy content of these waste products are lowered further by bacteria acting on it. Each genus of bacteria in turn lives on the residues or sewage of the higher order species, until such time, no other organism can derive any further benefit by this process. Biochemical degradation refers to the above reaction brought about by biolcgical agents to lower the energy contained in the molecular structure or to simplify the molecular structure. The stabilized end product is the fertilizer from which crops derive nutrients to complete the organic cycle.

The above organic growth and decay occur under two different sets of conditions. One proceeds when a sufficient quantity of free atmospheric oxygen is available for biodegradation by organisms in the water. This process is referred to as "aerobic decomposition". The other process is "anaerobic decomposition" in which organisms in the water release oxygen from other chemical compounds for the stabilization of organic matter. The energy released by the bacteria under the aerobic conditions is about thirty times [26] that is available to the bacteria

under anaerobic condition. This accounts for faster stabilization by the former rather than by the latter. In addition, the anaerobic end products are not stable. Thus they are still subject to subsequent aerobic digestion when the conditions are right. For the complete description of the two processes refer to figure 1 [26].

From the municipal sewage the return water, carrying man's ground garbage, human feces, and urine, will be subject to the right hand half of the different cycles when the unstable organic compounds are rendered into the river. The amounts of these unstable compounds can be measured by a quantity known as the biochemical oxygen demand (BOD) which is usually defined as the amount of oxygen required by bacteria while stabilizing decomposable organic matter under aerobic conditions. The ultimate BOD of the added sewage is the total amount of oxygen required to reduce the organic matter to stable compounds. Sometimes the oxygen demand proceeds at rates slower than natural reaeration of the receiving water, while at other times it far exceeds these rates. These levels of the dissolved oxygen (DO) are derived directly from the air or indirectly through photosynthetic processes of aquatic plants. It should be obvious that BOD and DO are the prime indicators of the quality of water in a river after organic sewage is dumped into it.

For perspective the BOD from all municipal sources throughout Canada for the twentieth century is presented in figure 2 [12]. It is an increasing curve since the amount depends on the population and the standard of living of the country. The present state of technology can limit the final concentration of BOD added to a river by secondary

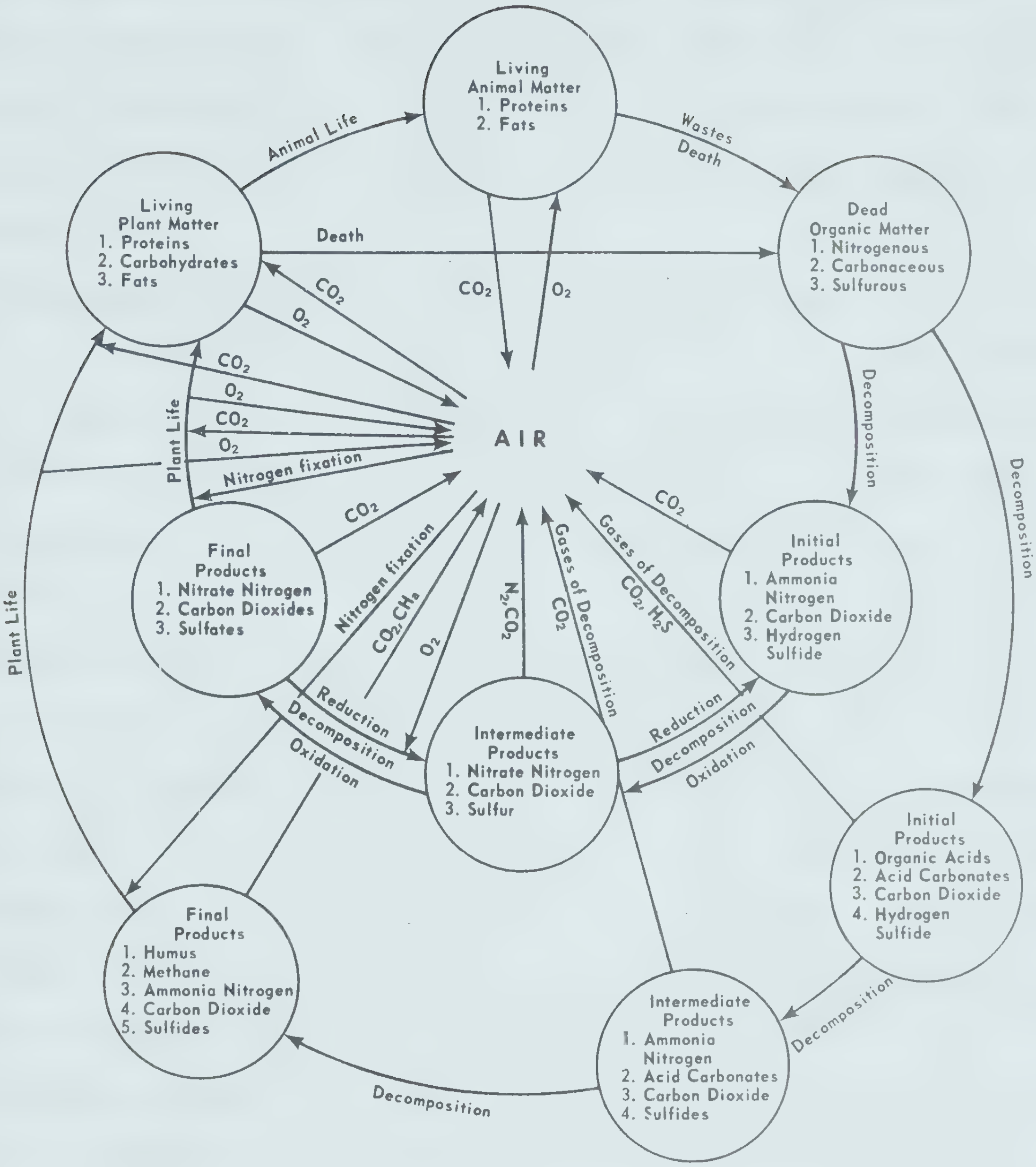

FIGURE 1: Nitrogen, Carbon, and Sulfur Cycle in Aerobic and Anaerobic Decomposition

sewage treatment which achieves a ninety percent removal rate and by tertiary sewage treatment which removes an additional five to ten percent. Even with the tertiary system the concentration of the BOD dumped into a river is one to four and half percent of the initial value received by a treatment plant.

1.3 Application of Modern Control Theory to Water Pollution

In the past few years there has been a number of papers published on the application of control/system theory to the problem of adding waste to a river [1,6,7,8,9,22,23,24]. This research has been concerned with the relationship between the concentration of DO and BOD caused by decomposing organic matter in a waterway. The problem is formulated as an optimal control problem in which the DO and BOD are used as controls to minimize a performance criterion according to calculus of variation techniques. In several of the papers [1,6,22,23] the objective functional is formulated either as a single index or as a linear combination of several performance indices using weighting factors to relate them. These weighting factors are arbitrarily specified a priori in the papers by Tarassov, Perlis and Davidson [23] and by Fan, Nadkarni and Erickson [6]. This approach has a drawback since it implies that a known relationship exists between the different performance indices used by them, which may not always be the case.

1.4 Scope of Thesis

The intent of research reported in this thesis is to determine the optimum dumping control, i.e. the addition of organic waste, and the optimum aeration control, i.e. the addition of oxygen by artificial

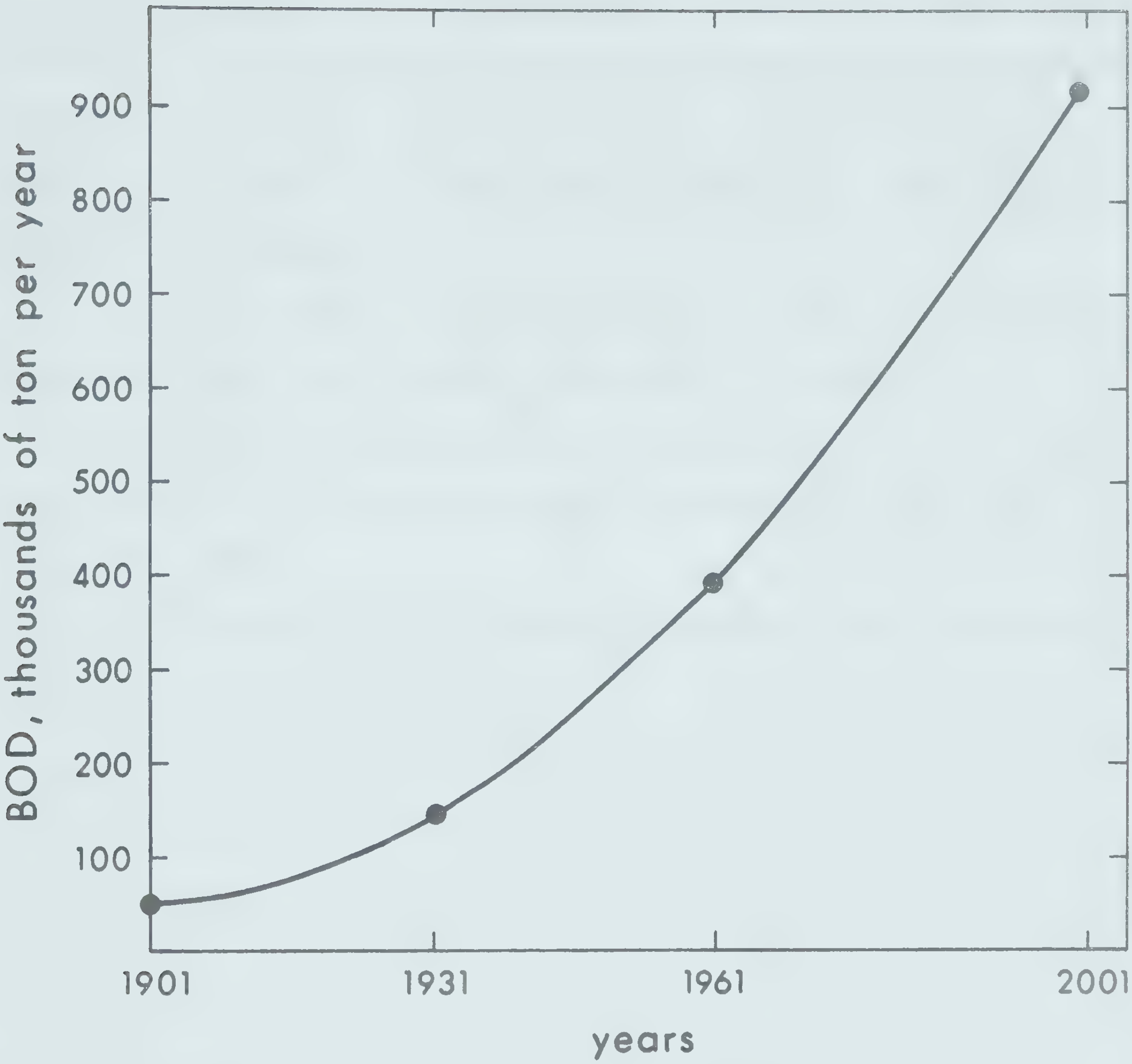

FIGURE 2: BOD from Municipal Sources throughout Canada

means, in terms of the weighting factors for all possible objective functionals relating the selected performance indices. In this manner more insight can be gained into the behaviour of the system.

In chapter two, a lumped parameter model in state space form is derived. The concentration of BOD and the deviation of the concentration of the actual DO from its saturation value are chosen as the state variables. This model is compared with a bilinear model which is also derived in this chapter.

Chapter three discusses the criteria for optimum water quality. Performance criteria and objective functionals are formulated.

In chapter four, optimization of water quality is carried out using the multi-cost optimization approach developed by Salama and Gourishankar [20].

In chapter five, the results are summarized and conclusions stated.

CHAPTER (2)

MATHEMATICAL MODEL

2.1 Physical Problem

The system to be modeled consists of a section of a river with a continuously-treated municipal waste discharged into it. (See figure 3). The addition of artificial aeration along the section can be included, if necessary. Starting with a distributed parameter model used by earlier investigators, it will be shown that by making appropriate simplifying assumptions, a lumped parameter linear model in state space form consisting of two first order equations will be derived. A bilinear model will also be described.

2.2 Distributed Parameter Model

From the previous chapter we have determined that it would be advantageous if we could determine the demand and the supply of DO at any point in a section of a river. In general, the movement and reactions of pollutants being transported in a river, can be described by the conventional diffusion equation [19] having the form

$$\frac{\partial c}{\partial t} = D_x \frac{\partial^2 c}{\partial x^2} + D_y \frac{\partial^2 c}{\partial y^2} + D_z \frac{\partial^2 c}{\partial z^2} - V(x,t) \frac{\partial c}{\partial x} - r(c) \qquad (2\text{-}1)$$

where c is the concentration of the pollutant

t is the time

x is the longitudinal distance along the river

y is the width of the river

z is the depth of the river

D_j is the dispersion coefficient of the river ($j=x,y,z$)

$r(c)$ is the rate of pollutant decay

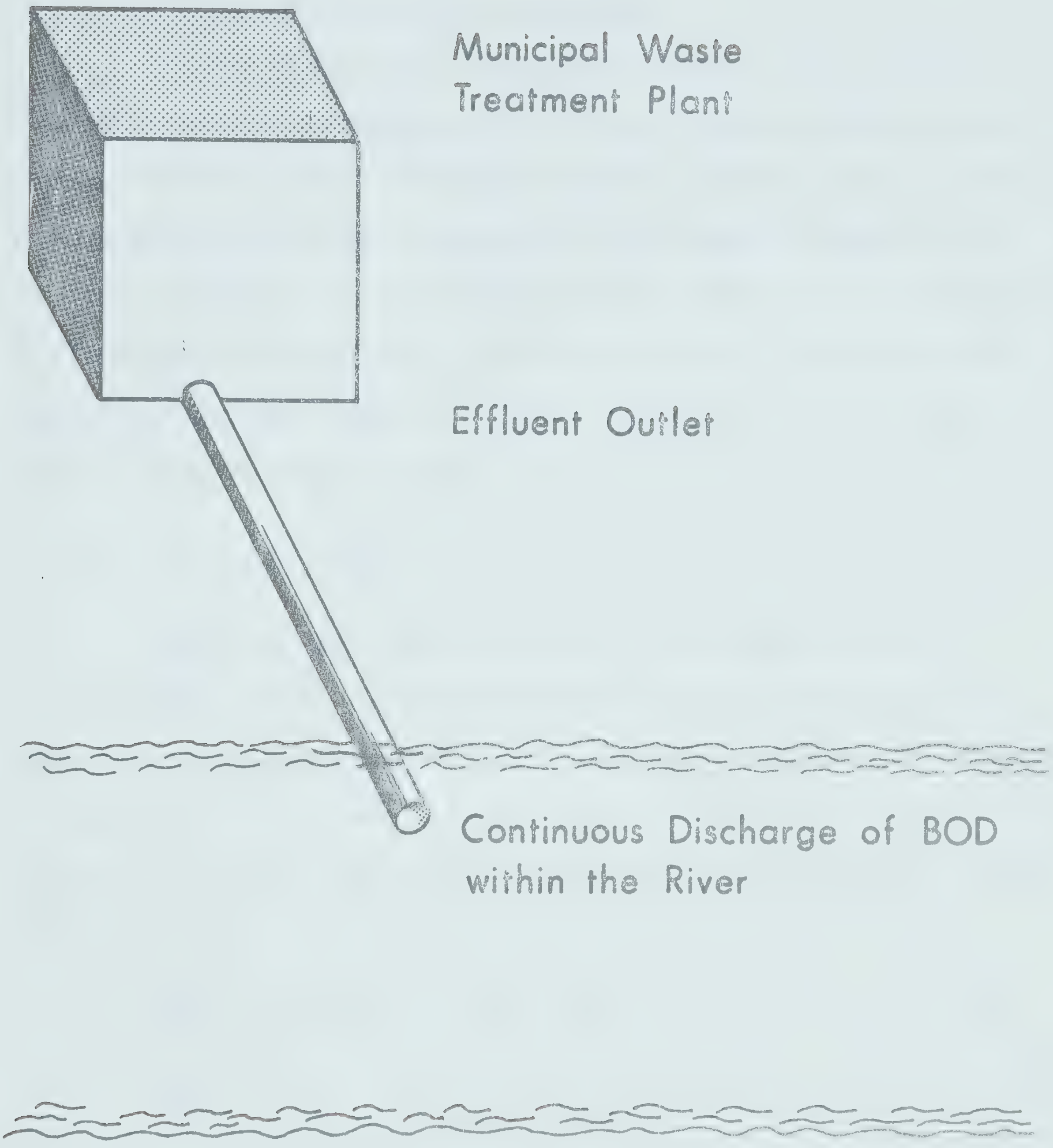

FIGURE 3: Schematic of River Section

$V(x,t)$ is the velocity of the river.

If we recognize that the width and depth are small compared to the length under consideration, then it can be assumed complete mixing is achieved within a short distance downstream. Now the terms involving y and z disappear and the concentration of the mixture depends on the downstream distance from the beginning of the river section. The effects of dispersion along the length of the river are small compared to the rate of transportation downstream due to the velocity of the river. Equation (2-1) becomes [5,11,23],

$$\frac{\partial c}{\partial t} = -V(x,t)\,\frac{\partial c}{\partial x} - r(c) \tag{2-2}$$

Using the above general equation, two equations can be written to describe the level of pollution due to organic waste. The first is concerned with the BOD due to the oxygen required by the bacteria to stabilize the decomposable organic waste under aerobic conditions. The second equation gives the DO derived from the air and the aquatic plants. The equations are

$$\frac{\partial B(x,t)}{\partial t} = -K_r(x,t) - V(x,t)\,\frac{\partial B(x,t)}{\partial x} \tag{2-3}$$

$$\frac{\partial D(x,t)}{\partial t} = -K_d(x,t)\;B(x,t) - V(x,t)\,\frac{\partial D(x,t)}{\partial x} + K_a(x,t)\;[D_s(x,t) - D(x,t)] \tag{2-4}$$

where D is the DO concentration

B is the BOD concentration

K_a is the reaeration coefficient

K_r is the BOD removal coefficient

K_d is the deoxygenation coefficient

D_s is the saturation level of the DO.

Next we consider a section of a river which has uniform properties. In other words, the above coefficients are constant with respect to time and distance. Equations (2-3) and (2-4) become

$$\frac{\partial B(x,t)}{\partial t} = -K_r \, B(x,t) - V(x) \frac{\partial B(x,t)}{\partial x} \tag{2-5}$$

$$\frac{\partial D(x,t)}{\partial t} = -K_a \, D(x,t) - K_d \, B(x,t) + K_a D_s - V(x) \frac{\partial D(x,t)}{\partial x} \tag{2-6}$$

Equations (2-5) and (2-6) constitute a distributed parameter model of the waste disposal system. This model has been used by some of the earlier investigators [18,22,23].

2.3 Lumped Parameter Model

If the concentration of BOD from the dumping source is constant with respect to time, we get the following lumped parameter model of the system [1,4,9,18,20],

$$\frac{dB(x)}{dx} = -\frac{K_r}{V} B(x) \tag{2-7}$$

$$\frac{dD(x)}{dx} = -\frac{K_d}{V} B(x) + \frac{K_a}{V} [D_s - D(x)] \tag{2-8}$$

describing the concentration of the BOD and DO for a section of a river.

As can be seen from equations (2-7) and (2-8), we have not taken into consideration the addition of organic wastes in terms of BOD. Since the levels of BOD and DO along a section depend on the inflow of organic waste, we define $u_1(x)$ to be the control function representing this input. Now the mathematical model becomes

$$\frac{dB(x)}{dx} = -\frac{K_r}{V} B(x) + \frac{u_1(x)}{V} \tag{2-9}$$

$$\frac{dD(x)}{dx} = -\frac{K_d}{V} B(x) + \frac{K_a}{V} [D_s - D(x)] \tag{2-10}$$

Let us now cast the model in matrix form using the following state variables representation,

$$x_1(y) = B(x)$$

$$x_2(y) = D_s - D(x)$$

where y replaces x, the distance downstream for a section of a river, to avoid confusion in the notation.

Thus,

$$\begin{bmatrix} \dot{x}_1(y) \\ \dot{x}_2(y) \end{bmatrix} = \begin{bmatrix} -B & 0 \\ C & -A \end{bmatrix} \begin{bmatrix} x_1(y) \\ x_2(y) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u_1(y)] \tag{2-11}$$

Initial Conditions: $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$

where $A = K_a/V$, $B = K_r/V$, $C = K_d/V$ and $u_1(y) = u(x)/V$.

A block diagram representation of (2-11) is shown in figure 4.

2.4 Bilinear Model

A bilinear model is now set up which takes into account the effect of DO on the rate at which BOD declines along a section of a river. The form of the bilinear model is

$$\frac{dB(x)}{dx} = -\frac{K}{V} D(x) \cdot B(x) \qquad (2\text{-}12)$$

$$\frac{dD(x)}{dx} = -\frac{Kd}{V} B(x) + \frac{Ka}{V} [D_s - D(x)] \qquad (2\text{-}13)$$

The only difference between the two models lies in the equation for the rate of change of BOD. The constant K_r, the BOD removal coefficient, has now been replaced by the expression K D(x), a constant multiplied by the level of DO along a river. This system could be representative of the aerobic decomposition since the BOD level would remain constant when there is no free oxygen present in a section of a river. Then another relationship would have to be developed for the case of anaerobic decomposition.

Both systems, the lumped parameter model and the bilinear model, were solved with the aid of an analog computer (See Appendix I) under identical conditions, while the values of K_r and K were varied over a specified range. The results are shown in figures 5 and 6. The shapes are similar for both models which leads to the following relationship between the two coefficients of BOD removal,

$$K \sim 1.41 \cdot \frac{K_r}{D_s} \qquad (2\text{-}14)$$

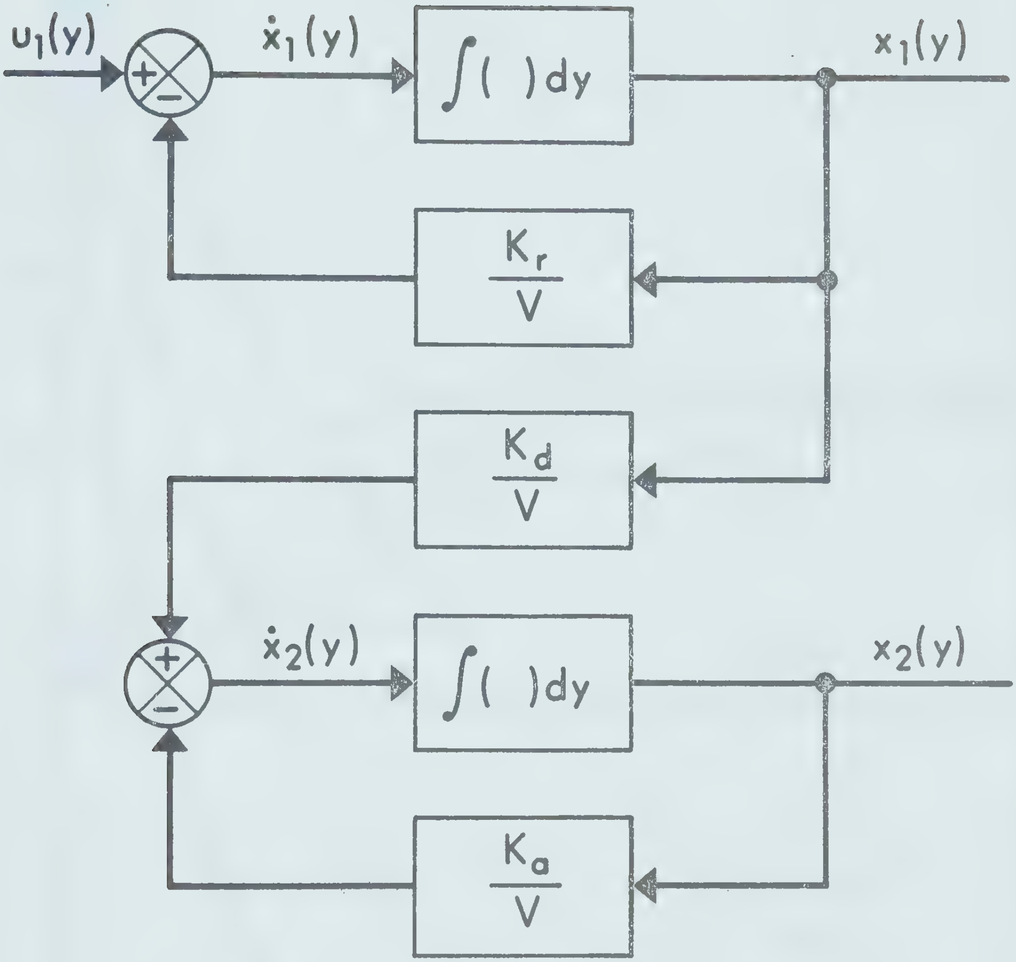

FIGURE 4: Block Diagram of System

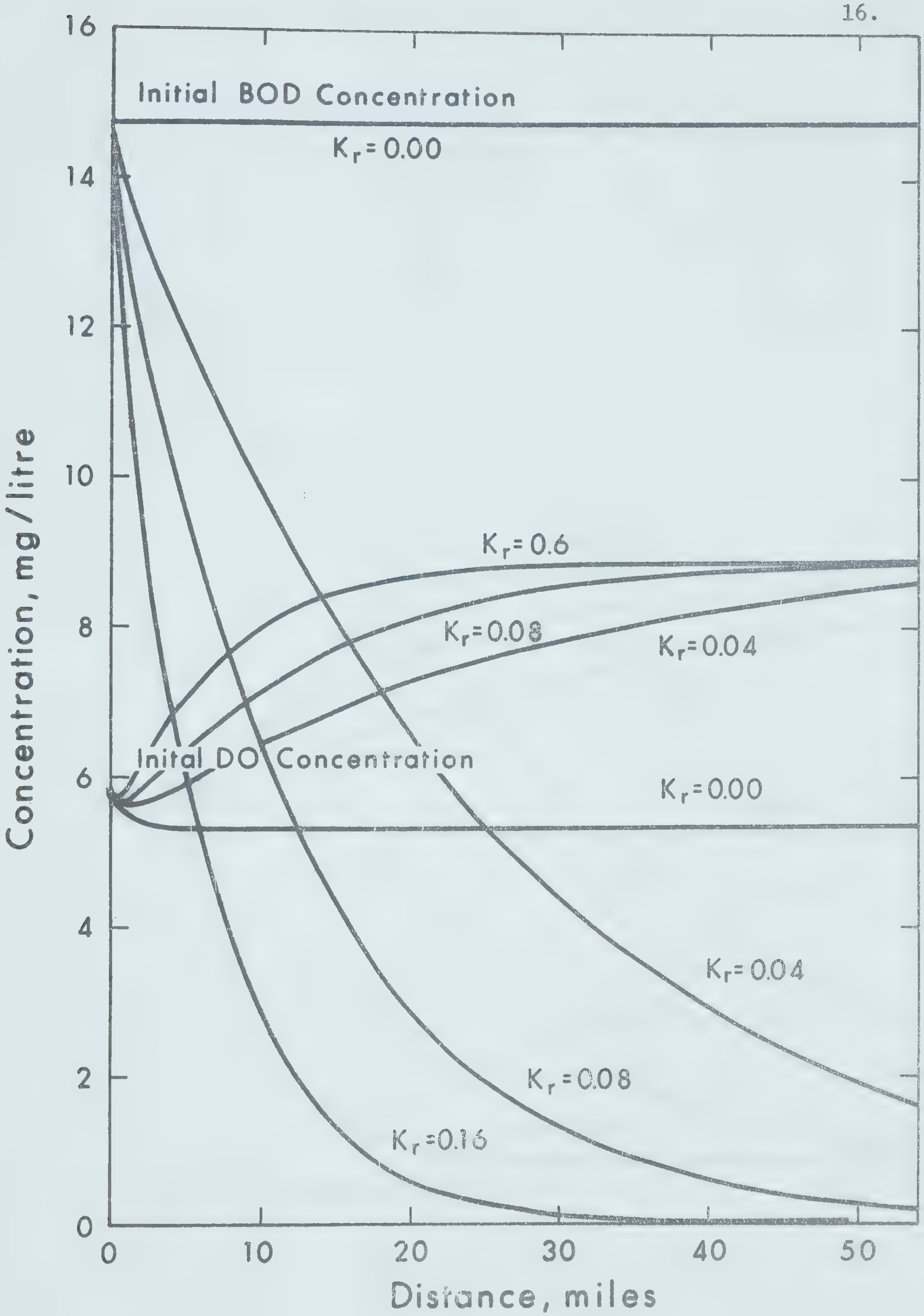

FIGURE 5: Lumped Parameter Model - BOD and DO Profiles

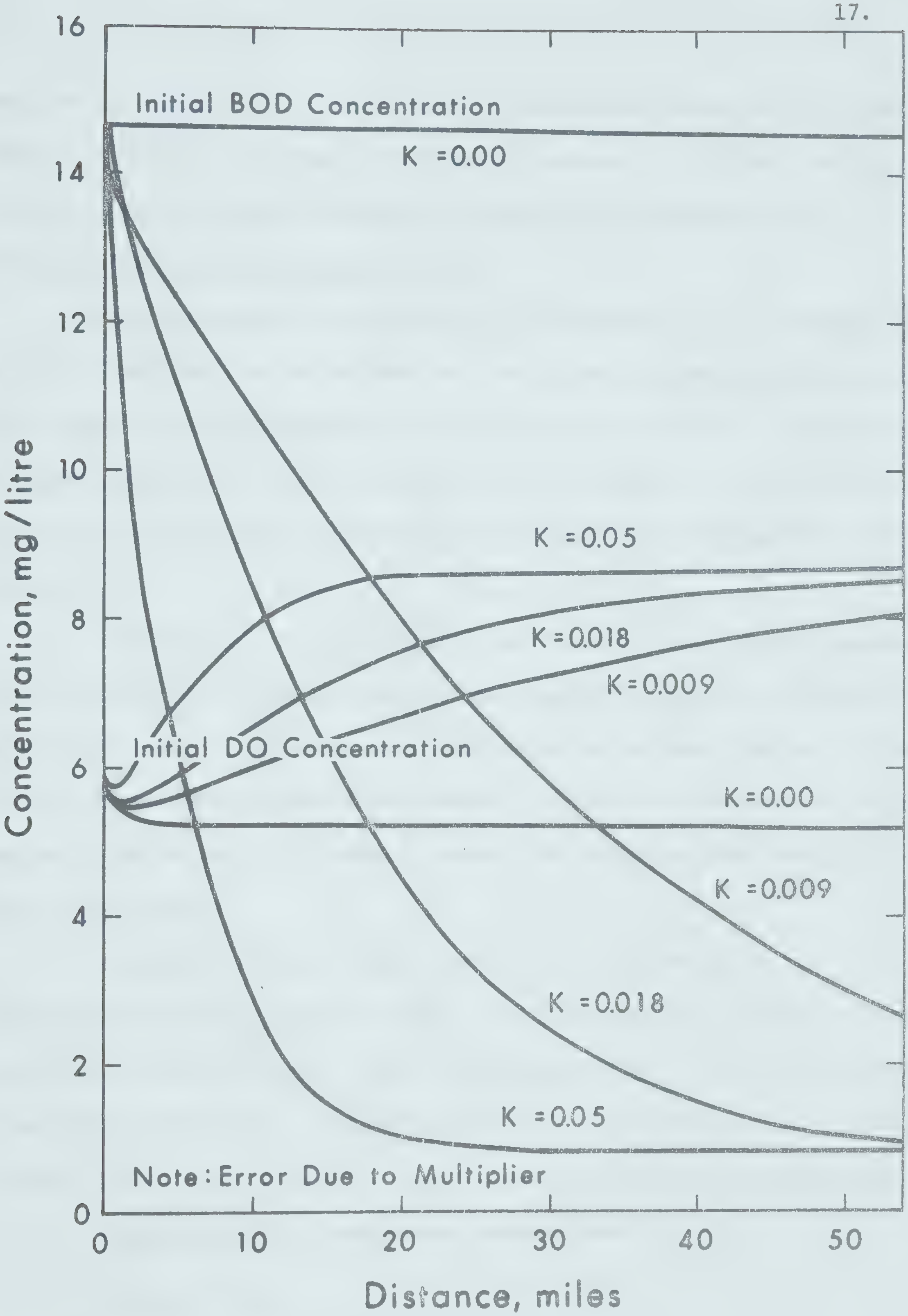

FIGURE 6: Bilinear Model - BOD and DO Profiles

when the errors due to the multiplier of the analog computer are ignored. Equation (2-14) is verified by the analog computer in figure 7 and by a ditigal computer using the Runge-Kutta Method (See Appendix II).

2.5 Comparison of the Different Models

The mathematical model used by Fan, Nadkarni, and Erickson [6] in their investigation is the same as the lumped parameter model derived here, except for the dispersion term which is disregarded. For our case the major mechanism of transportation for a pollutant is the velocity of a river rather than the much slower physical process of dispersion through a liquid.

The difference between the lumped and the distributed parameter models is that the former allows for the determination of the optimum controls in terms of only distance while the latter can give the optimum controls in terms of time and distrance. In practice however, a control with only one degree of freedom is easier to implement than one with two degrees of freedom.

We shall conclude this chapter with a discussion of the controllability of the model derived. If the system is completely state-controllable then for any y_o, each initial state $\underline{x}(y_o)$ can be transferred to any final state $\underline{x}(y_f)$ in a finite distance y_f which is greater than or equal to y_o. Let us outline the fundamental theorem concerning this,

Theorem I[17]: A continuous system described by

$$\dot{\underline{x}} = \underline{A}\ \underline{x} + \underline{B}\ \underline{u}$$

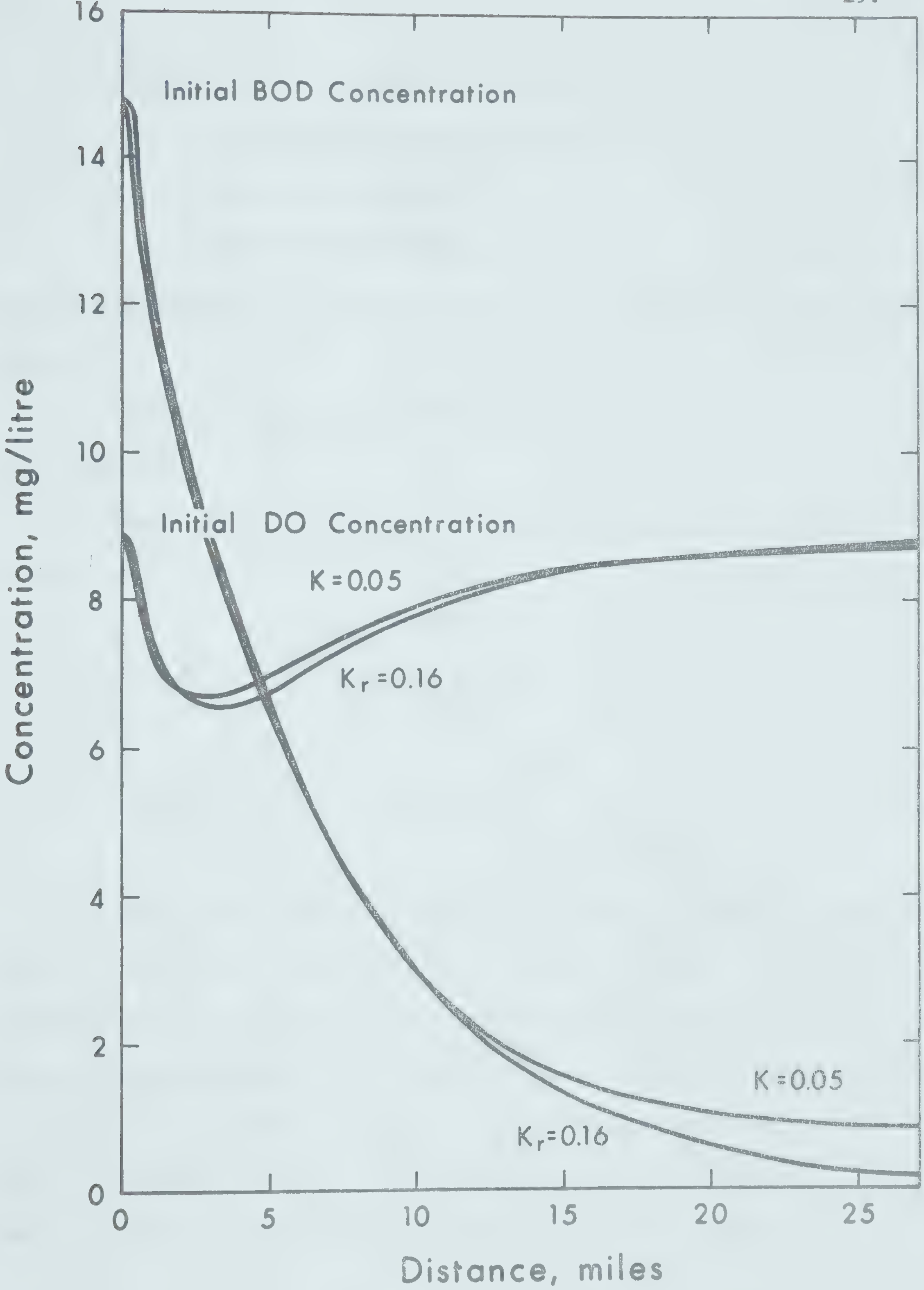

FIGURE 7: Comparison of Lumped Parameter and Bilinear Model Profiles

where $\underline{x}$ is an n-dimensional vector

$\underline{u}$ is an r-dimensional vector

$\underline{A}$ is an nxn matrix

$\underline{B}$ is an nxr matrix.

This is completely state-controllable if and only if the composite nxnr matrix,

$$\underline{P} = [\underline{B} \mid \underline{A}\,\underline{B} \mid \text{---} \mid \underline{A}^{n-1}\,\underline{B}]$$

is of rank n.

Returning to our system to check the rank of the matrix $\underline{P}$, we have

$$\underline{P} = [\underline{B} \quad \underline{A}\,\underline{B}] = \begin{bmatrix} 1 & -K_r/V \\ 0 & +K_d/V \end{bmatrix}$$

$$\text{where } B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \quad \underline{A} = \begin{bmatrix} -K_r/V & 0 \\ +K_d/V & -K_a/V \end{bmatrix}$$

The rank of the matrix $\underline{P}$ is two, hence the system is completely state-controllable. It may be noted at this time, that in case the coefficients K_r and K_d take on a range of values, the rank of matrix $\underline{P}$ remains constant except for K_d equal to zero. If this situation did hold true then the biochemical oxygen demand would have no effect on the dissolved oxygen level in a section of a river. This would be a contradiction with regards to the formulation of the problem.

CHAPTER (3)

CRITERIA FOR OPTIMUM WATER QUALITY

3.1 General Objectives of Water Quality Management

In general, all physical problems have the feature that, while at least one solution exists, there are usually an infinite number of solutions possible. The aim of the optimization becomes the selection, out of the multiplicity of potential solutions, of that solution which is the best with respect to some well-defined index of performance. This is true of water quality management problems also. In the past a number of papers have defined goals to be met, in this regard Bramhall and Mills [3] for instance have studied the effect of public policies on the benefits and costs to achieve desired water quality. Brown and Mar [4] have determined optimum standards to be measured against alternative water quality management policies.

To establish the criteria for optimum water quality the appropriate performance indices to generate the necessary objective functionals have to be selected for the system under consideration. Many performance indices could be examined to guide the determination of the best conditions under which an organic waste disposal system should be operated. In practice however it is not an easy task to specify or formulate performance indices out of their technical and economical contexts. The reason for this is the freedom of choice involved and the influence this choice has on other parameters in the problem. One basic performance index is the cost of BOD removal. Liebman and Lynn [14] have used this performance index subject to DO constraints in their study of water quality management of the Willamette

river in Oregon, U.S.A. That other choices of performance indices are possible can be demonstrated by a brief review of some more of the earlier investigators in the field of water resource management.

Fan, Nadkarni and Erickson [6] have divided a river into several segments and have proposed a cost benefit function on a segmented basis of the form:

$$S = \sum_{i=1}^{n} (\alpha_i C_{Ti} + \beta_i C_{Wi} - BF_i) \tag{3-1}$$

where C_{Ti} is the cost of waste treatment located at the end of the ith segment of the river under consideration for water quality management

C_{Wi} is the cost of water treatment located at the end of the ith segment

BF_i is the benefit function for the ith segment

α_i and β_i is the nonexistence (0) or the existence (1) of the ith facility.

The cost of waste treatment depends on several factors such as plant size, type of pollutant, degree of treatment, etc. Fan et al have suggested the typical curves for BOD treatment as shown in figure 8(a). For purposes of mathematical analysis and optimization a polynomial function which fits the curve is generated to give one performance index. Similarly, another polynomial function is fitted to the data for the total cost of purifying water for consumption verses the level of BOD in the water coming into the plant for another index (See figure 8(b)).

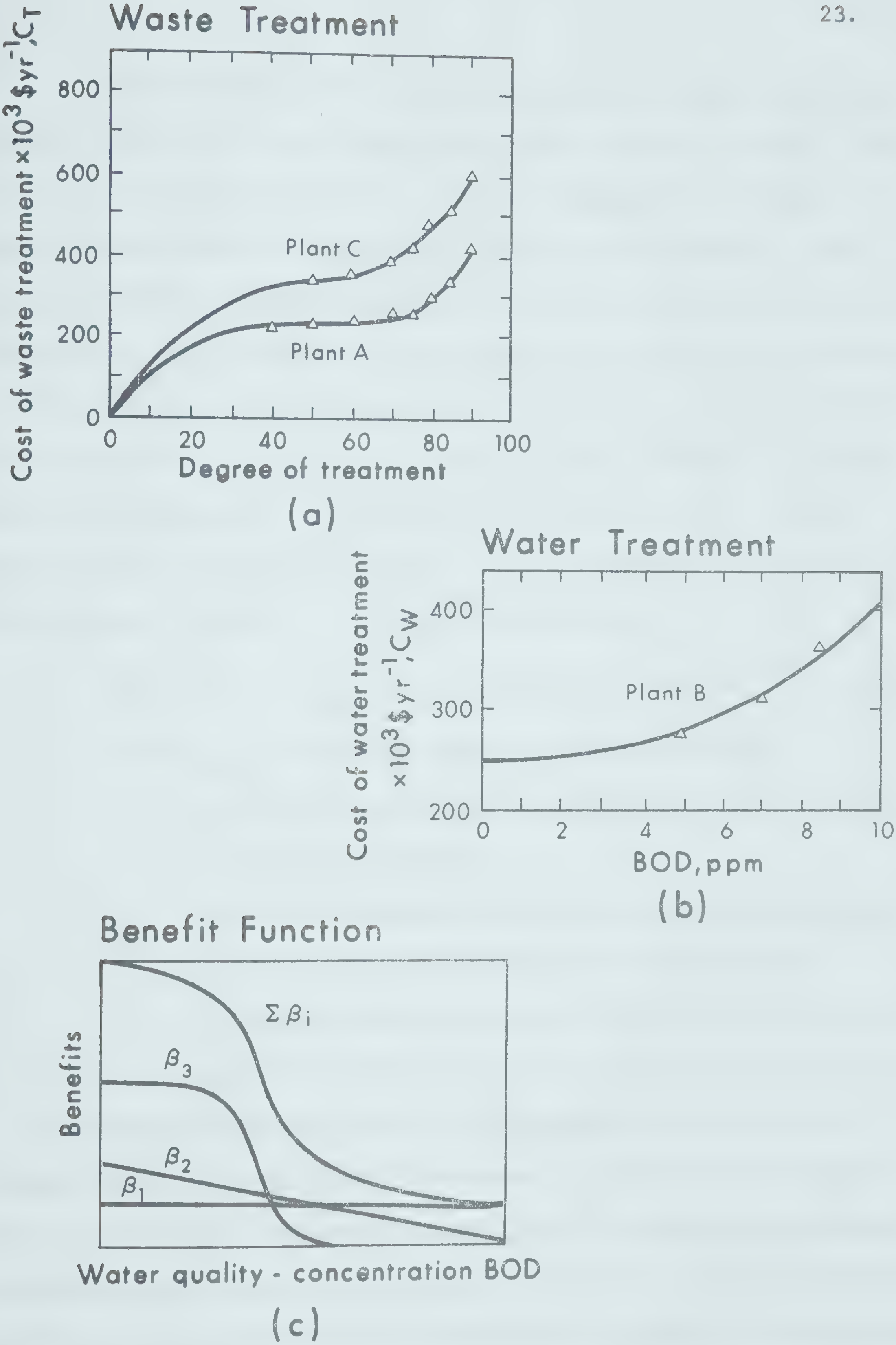

FIGURE 8: Cost of Waste Treatment, Cost of Water Treatment and Benefit Function

The remaining term BF_i in equation (3-1) reflects the price society is willing to pay to maintain a certain quality of water. While exact values for such a function are almost impossible to estimate, qualitive descriptions of the benefit function are possible. Brown and Mar [4] have suggested typical curves which are shown in figure 8(c). These curves are divided into three types, "(1) those that are independent of water quality, such as navigation, (2) those that decrease linearly with water quality, such as direct water use for consumption, and (3) those that decrease at a given threshold of water quality, such as aesthetic and recreational benefits". The mathematical representation of the benefit function is formed as the following:

$$BF_i = p_i - q_i \ (L_{Ai} - L_i) + v_i \exp \left[-w_i \ (L_i/L_{oi})\right] \qquad (3\text{-}2)$$

where p_i is a benefit independent of water quality

q_i is the gradient for benefits diminishing linearly with water quality

L_{Ai} is the maximum BOD concentration in the ith segment

L_i is the BOD concentration in the ith segment

v_i is a parameter for recreational and aesthetic benefits

L_{oi} is a parameter for recreational and aesthetic benefits.

Approaching the water quality management problem from an economic point of view, Boyd [1] determined a water quality management policy which seeks to maximize the value of a river's pollutant disposal and water quality services with respect to limiting pollutant discharges and/or increasing the river's waste assimilation capacity. The allocation of these services depends on the maximization of the net benefit, defined

by

$$B = \sum_{i=1}^{n} P_{zi} Z_i + \sum_{j=2}^{n} P_{qj} Q_j - C(K)$$

$$- \sum_{j=2}^{n} \lambda_j [Q_j - Q(Z_1, \ldots, Z_{j-1}, K)] \tag{3-3}$$

where B is the net benefit derived from the river's services

Z_i is the pollutant disposal services consumed at location i

Q_j is the water quality services enjoyed at location j

K is the waste assimilation capacity investment

$C(K)$ is the cost of K

λ is the Lagrangian multiplier

P_{zi} is the marginal product of Z_i evaluated at equilibrium

P_{qi} is the marginal product of Q_i evaluated at equilibrium.

To determine the specific relationship for the above performance indices, Boyd used equations (2-7) and (2-8) along with experimental data from the Potomac estuary below Washington, D.C.. The water quality services enjoyed, Q, is given by the difference between the saturation level of DO and the DO deficit which is the DO level of the river. By using table 1, the waste assimilation capacity, K, for two types of reoxygenation devices give the cost, $C(K)$, that would be incurred to offset given levels of pollutant discharge to maintain a minimum monthly mean of 4mg/ℓ of dissolved oxygen. Another set of data from the Polomac estuary is presented in table 2, which gives the flow requirements for given oxygen targets, waste loads and temperatures. From this table, the pollutant disposal services consumed, Z, is considered to be the

Table 1. Estimated Costs for Two Types of Aeration Devices, Potomac Estuary (Boyd)

UOD* Offset lb/day	Cost, $\$.10^6$ Low	Cost, $\$.10^6$ High	Cost/UOD $/(lb/day)	ΔCost/ΔUOD $/(lb/day)	Type
30,000	2.2	12.0	400		Diffused Reoxygenation
60,000	3.3	18.0	300	200	
90,000	4.6	27.0	300	300	
120,000	5.9	35.0	292	267	
30,000	4.4	9.2	307		Mechanical Reoxygenation
60,000	7.7	17.0	283	260	
90,000	3.9	25.0	278	267	
120,000	4.7	32.0	267	233	

*UOD, Ultimate Oxygen Demand

Table 2. Flow Requirements for Given Oxygen Target Waste Load, and Temperature, Potomac Estuary (Boyd)

UOD* $10^3 \cdot$ lb	Flow cfs	$\frac{\Delta F}{\Delta UOD}$	$\frac{F}{UOD}$	Conditions
140 200 260	500 2,200 6,000	28.3 63.3	3.6 11.0 23.1	DO = 4 ppm T = 28 C
140 200 260	800 3,500 8,000	45.0 75.0	5.7 17.5 30.8	DO = 4 ppm T = 29 C
140 200 260	1,100 4,800 8,000	61.7 53.3	7.9 24.0 30.8	DO = 4 ppm T = 30 C
140 200 260	500 1,500 4,000	16.7 41.7	3.6 7.5 15.4	DO = 3 ppm T = 30 C

*UOD, Ultimate Oxygen Demand

waste load added to the river while the flow rate, F (flow augmentation plus natural flow), is used to maintain a certain minimum DO concentration. To determine the marginal products, P_z and P_q, the concentration of DO, Q, is related to the flow rate, F, of the river then using the following they can be derived,

$$\frac{P_z}{P_q} = - \frac{\partial Q/\partial F}{\partial Z/\partial F} = - \frac{\partial Q}{\partial Z} \tag{3-4}$$

3.2 Performance Criteria for Distributed Parameter Model

Using a distributed parameter model given by equations (2-5) and (2-6), Tarassov, Perlis and Davidson [23] applied optimization theory for determining an artificial in-stream aeration policy for polluted rivers. They used an objective functional of the form:

$$J = c_1 \int_0^{t_f} \int_0^{x_f} [6.00 - C(t,x)]^2 \, dxdt + c_2 \int_0^{t_f} \int_0^{x_f} [u(t,x)]^2 \, dxdt \tag{3-5}$$

where $C(t,x)$ is the actual DO profile in a section of a river

$u(t,x)$ is the magnitude of the distributed artificial aeration from the in-stream aerator

c_1 and c_2 are the weighting factors.

The objective functional is a linear combination of two quadratic performance indices. The first performance index represents the difference between a specified value of DO level and the actual

DO profile. Minimization of this index will prevent the excessive deviation of the actual DO from the value of 6.00 mg/litre which is an assumed water quality standard to maintain the fish life and the aerobic conditions within a river. The second quadratic performance index represents a measure of the total control effort that must be expended to achieve the optimum DO profile. It is to be noted here that the values of the weighting factors are chosen prior to solving for u(t,x).

In a similar paper Perlis and Cook [18] have used the same mathematical model as in [23] but modified to include two controls, one an artificial aeration control and the other an effluent control. The objective functional is correspondingly modified as,

$$J = L_1 \int_{t_o}^{t_f} \int_{x_o}^{x_f} (VSP_1 - V_1)^2 \, dxdt + L_2 \int_{t_o}^{t_f} \int_{x_o}^{x_f} (U_1)^2 \, dxdt$$

$$+ C_1 \int_{t_o}^{t_f} \int_{x_o}^{x_f} (VSP_2 - V_2)^2 \, dxdt + C_2 \int_{t_o}^{t_f} \int_{x_o}^{x_f} (U_2)^2 \, dxdt \qquad (3\text{-}6)$$

where VSP_1 is the specified BOD level

VSP_2 is the specified DO level

V_1 is the actual BOD level

V_2 is the actual DO level

U_1 is the dumping control

U_2 is the aeration control

L_1, L_2, C_1 and C_2 are the weighting factors.

Thus equation (3-6) is an expanded form of equation (3-5) to handle the problem of determining a management policy for the controlled dumping of effluents as well as artificial aeration. The values of the four weighting factors, L_1, L_2, C_1 and C_2 are chosen individually prior to proceeding with the minimization of the objective functional.

3.3 Performance Criteria for Lumped Parameter Model

In this section we shall formulate a general performance index for the lumped parameter state space model. This will reflect the various demands placed on a section of a river due to the addition of organic waste. In this task we shall use cost functionals proposed by earlier investigators for specific purposes. If a certain quantity of BOD is released from a waste treatment plant, there will be a cost associated with the processing of it. A typical relationship between the cost of treatment and the level of BOD expelled from a given plant is shown in figure 9(a). Such a relationship has been proposed by several investigators [6,10,15,25]. Another cost function relates the cost to the environment and the concentration of BOD. Obviously the former increases with the latter. A typical plot is shown in figure 9(b) [16]. The cost to the environment refers to the lost recreational facilities and other activities due to pollution. If we combine figures 9(a) and 9(b), the result is a function which represents the cost of processing waste to lower the level of BOD and the cost of excess BOD on the environment and the society (See figure 9(c)). Loucks [15] constructed a similar convex function for the target allocation of water versus the quantity of water desired by a user, while Bibbero [2] has used previous material

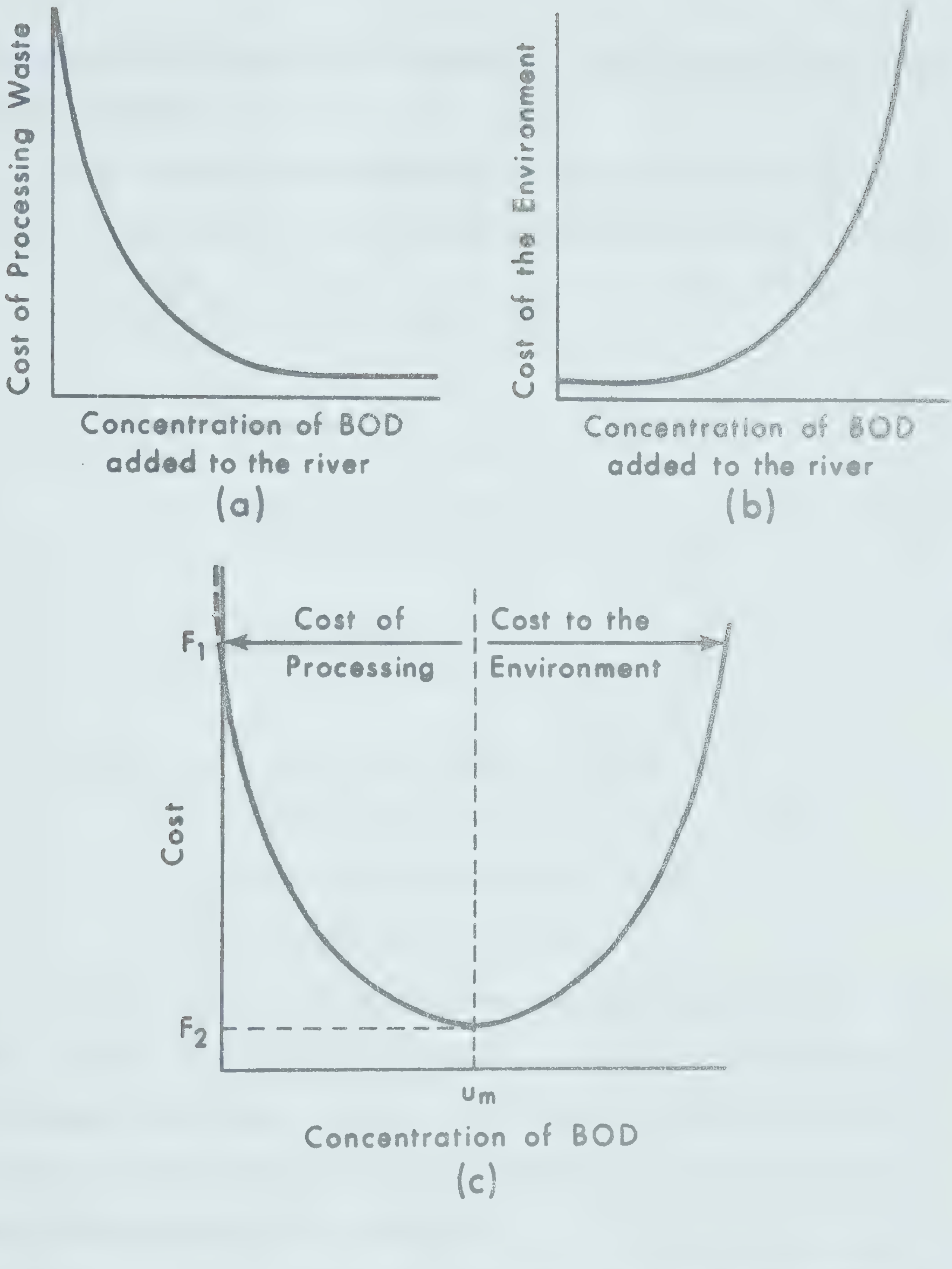

FIGURE 9: Cost of Processing Waste and Cost to the Environment

on cost-effective approach to establish the total societal cost for air pollution control.

By fitting a second degree polynomial to this function in figure 9(c), we define the performance index for a section of a river,

$$J = \int_{o}^{y_f} (a_1 - a_2\, u_1 + a_3\, u_1^2)\, dy \tag{3-7}$$

$$\left.\begin{aligned} \text{and}\quad a_1 &= F_1 \\ a_2 &= \left[\frac{2[F_1 + F_2\,(u_m - 1)]}{u_m}\right] - F_2 \\ a_3 &= \frac{F_1 + F_2\,(u_m - 1)}{u_m^2} \end{aligned}\right\} \tag{3-8}$$

where F_1, F_2 and u_m are defined in figure 8(c)

y is the distance along a section of a river

y_f is the end of a section of a river

u_1 is the BOD dumping control.

It is useful to formulate some more performance indices, which can be combined with that given in (3-8), to form complex objective functionals. When using a section of a river for the dumping of waste the effects on the next section should be minimal. This can be taken care of using terminal costs given by,

$$J_1 = x_1\,(y_f)\ , \tag{3-9}$$

the level of BOD at the end of a section of a river, and

$$J_2 = x_2\,(y_f)\ , \tag{3-10}$$

the difference between the saturation level of DO and the actual level of DO at end of a section of a river.

3.4 Linear Combination Objective Functional

Having formulated the three separate performance indices we shall next formulate a linear combination of the three indices. It will be shown in the next chapter that unlike earlier investigators, who used fixed values of the weighting factors, the optimum controls will be determined in terms of the weighting factors.

It will also be shown in the next chapter that once the linear combination problem has been solved the results can be used for any objective functional expressed as a function of the given indices and the optimal controls can be determined. The conventional methods requires that the optimization problem be solved for each objective functional separately which drastically increases the amount of work to be performed.

CHAPTER (4)

OPTIMIZATION OF WATER QUALITY

4.1 Introduction

Having derived an appropriate mathematical model in state space form for the river and having described the water quality management criteria by appropriate performance indices, the stage is now set for the optimization of water quality which is the subject of this chapter. In section 4.2, an introduction of the necessary control theory connected with the optimization of multi-cost system is provided. In section 4.3 a policy of water quality management is established for the controlled dumping of effluents (BOD) into a section of a river, subject to several different objective functionals. In section 4.4, the effects of artificial aeration is studied by introducing a vector control $[u_1 u_2]^T$ where u_1 represents the controlled dumping of effluents (BOD) and u_2 represents the addition of dissolved oxygen (DO). The optimization is carried out with respect to the same objective functionals used earlier with u_1 alone as the control.

4.2 Optimization Theory for Multi-Cost Systems

In this section the necessary theory related to the optimization of multi-cost systems will be summarized. For a more detailed discussion the reader is referred to the relevant literature [20].

Let the model of the system be represented by a set of n first order ordinary differential equations,

$$\left.\begin{aligned} \dot{\underline{x}}(y) &= \underline{f}\,[\underline{x}(y), \underline{u}(y)] \\ \underline{x}(0) &= \underline{x}_o \end{aligned}\right\} \qquad (4\text{-}1)$$

where $\underline{f}$ is a n-dimensional vector valued function continuous in $\underline{x}$ and $\underline{u}$

$\underline{x}$ is a n-dimensional vector representing the state of the system

$\underline{u}$ is a r-dimensional control vector of the system.

The N performance indices are expressible in the form,

$$J_K = g_K[\underline{x}\ (y_f)] + \int_0^{y_f} \ell_K[\underline{x}(\tilde{y}),\ \underline{u}(\tilde{y})]d\tilde{y},\ k=1,\ \ldots\ ,\ N \tag{4-2}$$

or reformulated as

$$\left.\begin{aligned} \dot{z}_K(y) &= \ell_K[\underline{x}(y),\ \underline{u}(y)] + \{\partial g_K[\underline{x}(y)]/\partial \underline{x}\}^T \cdot \\ &\qquad\qquad \underline{f}[\underline{x}(y),\ \underline{u}(y)] \\ z_K(0) &= g_K[\underline{x}(0)],\ k=1,\ \ldots\ ,\ N \\ z_K(y_f) &= J_K \end{aligned}\right\} \tag{4-3}$$

An objective functional is next defined as

$$\phi = \phi(z_1,\ \ldots\ ,\ z_N) \tag{4-4}$$

The control problem can be described by the following system of n + N = 1 equations,

$$\left.\begin{aligned}
&\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) \\
&\dot{z}_K = \ell_K(\underline{x}, \underline{u}) + [\partial g_K(\underline{x})/\partial \underline{x}]^T \cdot \underline{f}(\underline{x}, \underline{u}) = G_K(\underline{x}, \underline{u}) \\
&\dot{\phi} = [\partial\phi/\partial z_1] \cdot G_1(\underline{x}, \underline{u}) + \ldots + [\partial\phi/\partial z_N] \cdot G_N(\underline{x}, \underline{u}) \\
&\text{and} \\
&\underline{x}(0) = \underline{x}_o \\
&z_K(0) = g_K[\underline{x}(0)] \\
&\phi\,(0) = \phi[z_1(0), \ldots , z_N(0)] \\
&k=1, \ldots , N
\end{aligned}\right\} \quad (4\text{-}5)$$

To obtain the necessary conditions for optimality for the free right end point problem, that is $\underline{x}(y_f)$ free, with the final distance, y_f, fixed, the Hamiltonian is defined as

$$H = \underline{P}^T \cdot \underline{f}(\underline{x}, \underline{u}) + p_{n+1} \cdot G_1(\underline{x}, \underline{u}) + \ldots + p_{n+N} \cdot G_N(\underline{x}, \underline{u})$$
$$+ p_{n+N+1} \left[\frac{\partial\phi}{\partial z_1} G_1(\underline{x}, \underline{u}) + \ldots + \frac{\partial\phi}{\partial z_N} G_N(\underline{x}, \underline{u})\right] \quad (4\text{-}6)$$

where $\underline{P}$ is a n-dimensional vector. We get

$$\left.\begin{aligned}
&\dot{\underline{P}} = -\partial H/\partial \underline{x} \\
&\dot{p}_{n+K} = -\partial H/\partial z_K, \; k=1, \ldots , N \\
&\dot{p}_{n+N+1} = -\partial H/\partial \phi = 0
\end{aligned}\right\} \quad (4\text{-}7)$$

Using the fact that the right end point is free we have

$$\left.\begin{aligned} &\underline{P}(y_f) = 0 \\ &p_{n+K}(y_f) = 0, \ k=1, \ldots, N \\ &p_{n+N+1}(y_f) = -1 \end{aligned}\right\} \quad (4\text{-}8)$$

Now the expression for the Hamiltonian becomes

$$H = \underline{P}^T \cdot \underline{f}(\underline{x}, \underline{u}) + (p_{n+1} - \partial\phi/\partial z_1) \cdot G_1(\underline{x}, \underline{u}) + \ldots$$

$$+ (p_{n+N} - \partial\phi/\partial z_N) \cdot G_N(\underline{x}, \underline{u}) \quad (4\text{-}9)$$

It has been shown [20] that the quantities $(p_{n+K} - \partial\phi/\partial z_K)$, $k=1, \ldots, N$, are constants and hence can be treated weighting factors c_K. Using the second relationship of (4-8) the weighting factors become

$$c_K = -\partial\phi/\partial z_K \Big|_{y=y_f} \quad (4\text{-}10)$$

The Hamiltonian can now be rewritten in a simpler form as

$$H = \underline{\tilde{P}}^T \cdot \underline{f}(\underline{x}, \underline{u}) + \sum_{k=1}^{N} c_K \ell_K(\underline{x}, \underline{u}) \quad (4\text{-}11)$$

$$\left.\begin{aligned} &\text{where} \quad \underline{\tilde{P}}^T = \underline{P}^T + \sum_{k=1}^{N} c_K[\partial g_K(\underline{x})/\partial \underline{x}]^T \\ &\dot{\underline{\tilde{P}}} = -\partial H/\partial \underline{x} \\ &\underline{\tilde{P}}(y_f) = + \sum_{k=1}^{N} c_K[\partial g_K(\underline{x})/\partial \underline{x}] \Big|_{y=y_f} \end{aligned}\right\} \quad (4\text{-}12)$$

The procedure for the minimization of any objective functional, ϕ, consists of the following steps:

Step (1) Consider the optimization problem with the same system of equations and conditions as in the given problem but with a linear combination objective functional, $\sum_{k=1}^{N} c_K z_K$. Determine the optimum control, $\underline{u}^*$, as a function of the c_K's.

Step (2) Search for the optimum values of the c_K's which minimize the given ϕ. This is accomplished by searching for the location where the gradient of ϕ is zero either with respect to the weighting factor ratios or the performance indices. In other words the search is made either in the cost scales space or the weighting factor space.

Step (3) Determine the optimum control for the given ϕ by substitution of the optimum weighting factor ratios from step (2) into the expression for the control obtained in step (1).

4.3 Management of Water Quality by Controlled Dumping of Effluent (BOD)

As our first attempt at optimization of water pollution, we shall consider the problem of determining the optimal dumping of effluents for the following three cost scales or performance indices.

Let

$$J_1 = x_1(y_f)$$

represent the level of BOD at the end of a section of a river.

Let

$$J_2 = x_2(y_f)$$

represent the difference between the saturation level of DO and the

actual level of DO at the end of a section of a river, and let

$$J_3 = \int_0^{y_f} (a_1 - a_2u_1 + a_3u_1^2)\, dy$$

represent the cost of processing waste to lower the level of BOD and the cost of excess BOD on the environment and to society, such as the aesthetic and recreational benefits.

For the optimal control problem the system of equations is given by

$$\left.\begin{array}{ll}
\dot{x}_1(y) = -B\, x_1(y) + u_1(y) & x_1(0) = x_{10} \\
\dot{x}_2(y) = C\, x_1(y) - A\, x_2(y) & x_2(0) = x_{20} \\
\dot{z}_1(y) = 0 & z_1(0) = x_{10} \\
\dot{z}_2(y) = 0 & z_2(0) = x_{20} \\
\dot{z}_3(y) = a_1 - a_2u_1(y) + a_3[u_1(y)]^2 & z_3(0) = 0
\end{array}\right\} \quad (4\text{-}13)$$

We have to satisfy the following conditions:

(i) final conditions, $\underline{x}(y_f)$, are not fixed

(ii) the dumping control, $u_1(y)$, is unconstrained

(iii) the distance, y_f, is fixed

We shall consider the following objective functionals:

(i) $$\phi(z_1, z_2, z_3) = z_3 - (c_1/c_3)\, z_1 - (c_2/c_3)\, z_2 \quad (4\text{-}14)$$

(ii) $$\phi(z_1, z_2, z_3) = a_3z_1 \cdot z_2 + z_3 \quad (4\text{-}15)$$

$$\text{(iii)}\ \phi(z_1, z_2, z_3) = a_3(z_1^2 + z_2^2)/2 + z_3 \tag{4-16}$$

Solution of the Linear Combination Problem

The first step, namely, the solution of the linear combination problem is common for above three objective functionals. Using the linear combination objective functional, $c_1z_1 + c_2z_2 + c_3z_3$, the Hamiltonian becomes

$$H = c_3(a_1 - a_2u_1 + a_3u_1^2) - Bx_1p_1 + u_1p_1 + Cx_1p_2 - Ax_2p_2 \tag{4-17}$$

And

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = Bp_1 - Cp_2, \quad p_1(y_f) = c_1 \tag{4-18}$$

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = Ap_2 \qquad , \quad p_2(y_f) = c_2 \tag{4-19}$$

After solving the costate equations, the expression for the costate variables are

$$p_1 = c_1e^{B(y-y_f)} + \frac{c_2C}{(B-A)}\left(e^{A(y-y_f)} - e^{B(y-y_f)}\right) \tag{4-20}$$

$$p_2 = c_2e^{A(y-y_f)} \tag{4-21}$$

With no constraints on the control function the optimum dumping control takes on the following form,

$$u_1(y) = \frac{a_2}{2a_3} - K_1e^{B(y-y_f)} - K_2e^{A(y-y_f)} \tag{4-22}$$

where $$K_1 = \frac{1}{2a_3}\left[\left(\frac{c_1}{c_3}\right) - \frac{C}{(B-A)}\left(\frac{c_2}{c_3}\right)\right]$$

$$K_2 = \frac{1}{2a_3}\left[\frac{C}{(B-A)}\left(\frac{c_2}{c_3}\right)\right]$$

Substituting the optimum dumping control back into the system of equations (4-13), the BOD profile is given by

$$x_1(y) = x_{10}e^{-By} + \frac{a_2}{2a_3}\frac{(1-e^{-By})}{B} - K_1\frac{(e^{B(y-y_f)}-e^{-B(y-y_f)})}{2B} - K_2\frac{(e^{A(y-y_f)}-e^{-By-Ay_f})}{(B+A)} \tag{4-23}$$

The profile for the saturation level minus the actual level of dissolved oxygen is described by the expression

$$x_2(y) = x_{20}e^{-Ay} + \frac{C}{(B-A)}\left[x_{10}e^{-Ay} + \frac{a_2}{2a_3}\frac{(1-e^{-Ay})}{A} - K_1\frac{(e^{B(y-y_f)}-e^{-Ay-By_f})}{(B+A)} - K_2\frac{(e^{A(y-y_f)}-e^{-A(y-y_f)})}{2A} - x_1(y)\right] \tag{4-24}$$

Using equations (4-23) and (4-24) we can obtain the expressions for the performance indices as,

$$z_1 = x_{10}e^{-By_f} + \frac{a_2}{2a_3}\frac{(1-e^{-By_f})}{B} - K_1\frac{(1-e^{-2By_f})}{2B} - K_2\frac{(1-e^{-(B+A)y_f})}{(B+A)} \tag{4-25}$$

$$z_2 = x_{20}e^{-Ay_f} + \frac{C}{(B+A)}\left[x_{10}e^{-Ay_f} + \frac{a_2}{2a_3}\ \frac{(1-e^{-Ay_f})}{A}\right.$$

$$\left. - K_1\frac{(1-e^{-(B+A)y_f})}{(B+A)} - K_2\frac{(1-e^{-2Ay_f})}{2A} - z_1\right] \tag{4-26}$$

and

$$z_3 = \left[a_1 - \frac{a_2^{\ 2}}{4a_3}\right] y_f + a_3\left[K_1^{\ 2}\ \frac{(1-e^{-2By_f})}{2B}\right.$$

$$\left. + 2K_1K_2\frac{(1-e^{-(B+A)y_f})}{(B+A)} + K_2^{\ 2}\frac{(1-e^{-2Ay_f})}{2A}\right] \tag{4-27}$$

At this point, in order to point out the optimization procedure effectively we shall use the numerical values used by Perlis and Cook [18]. Let

$$\left.\begin{aligned} K_a &= 0.66/\text{day} \\ K_d = K_r &= 0.16/\text{day} \\ V &= 1.00\ \text{miles/day} \end{aligned}\right\} \tag{4-28}$$

In addition to the above values, the distance under consideration is chosen equal to two and one-half miles. Using these values, the three performance indices become

$$z_1 = 0.6703\ x_{10} + \frac{1}{a_3}\left[1.0303\ a_2 - 0.8605\left(\frac{c_1}{c_3}\right)\right.$$

$$\left. - 0.1053\left(\frac{c_2}{c_3}\right)\right] \tag{4-29}$$

$$z_2 = 0.1531\ x_{10} + 0.1920\ x_{20} + \frac{1}{a_3}\left[0.1338\ a_2 - 0.1053\left(\frac{c_1}{c_3}\right)\right.$$

$$\left. - 0.01666\left(\frac{c_2}{c_3}\right)\right] \tag{4-30}$$

$$z_3 = 2.5\left[a_1 - \frac{a_2^2}{4a_3}\right] + \frac{1}{a_3}\left[0.4303\left(\frac{c_1}{c_3}\right)^2 + 0.1053\left(\frac{c_1}{c_3}\right)\left(\frac{c_2}{c_3}\right)\right.$$

$$\left. + 0.00833\left(\frac{c_2}{c_3}\right)^2\right] \tag{4-31}$$

where

$$a_1 = F_1,\ a_2 = 2\ \frac{[F_1+F_2(u_m-1)]}{u_m} - F_2 \text{ and } a_3 = \frac{F_1+F_2(u_m-1)}{u_m^2}$$

as defined in section 3.3.

In reality, the cost of processing waste, F_1, approaches infinity as the BOD concentration is lowered to zero. This allows us to rewrite the above as

$$z_1 = 0.6703\ x_{10} + 2.0606\ u_m - \frac{1}{a_3}\left[0.8605\left(\frac{c_1}{c_3}\right)\right.$$

$$\left. + 0.1053\left(\frac{c_2}{c_3}\right)\right] \tag{4-32}$$

$$z_2 = 0.1531\ x_{10} + 0.1920\ x_{20} + 0.2676\ u_m$$

$$- \frac{1}{a_3}\left[0.1053\left(\frac{c_1}{c_3}\right) + 0.01666\left(\frac{c_2}{c_3}\right)\right] \tag{4-33}$$

$$z_3 = \frac{1}{a_3}\left[0.4303\left(\frac{c_1}{c_3}\right)^2 + 0.1053\left(\frac{c_1}{c_3}\right)\left(\frac{c_2}{c_3}\right) + 0.00833\left(\frac{c_2}{c_3}\right)^2\right] \quad (4\text{-}34)$$

Minimization of the First Objective Functional

Substituting the expression for z_1, z_2 and z_3 from (4-32) thru (4-34) into (4-14), we get

$$\begin{aligned}\phi &= z_3 - (c_1/c_3)\, z_1 - (c_2/c_3)\, z_2 \\ &= \frac{1}{a_3}\left[1.2908\left(\frac{c_1}{c_3}\right)^2 + 0.3159\left(\frac{c_1}{c_3}\right)\left(\frac{c_2}{c_3}\right) + 0.02499\left(\frac{c_2}{c_3}\right)^2\right] \\ &\quad - \left[0.6703\, x_{10} + 2.0606\, u_m\right]\left(\frac{c_1}{c_3}\right) - \left[0.1531\, x_{10} \right. \\ &\quad \left. + 0.1920\, x_{20} + 0.2676\, u_m\right]\left(\frac{c_2}{c_3}\right) \end{aligned} \quad (4\text{-}35)$$

To determine the optimum values of the weighting factor ratios, c_1/c_3 and c_2/c_3, which will minimize the objective functional, we take the gradient of ϕ and set it equal to zero to get

$$(c_1/c_3) = (-0.5084\, x_{10} - 2.0746\, x_{20} + 0.6312\, u_m)\, a_3 \quad (4\text{-}36)$$

$$(c_2/c_3) = (6.2765\, x_{10} + 16.9542\, x_{20} + 1.3645\, u_m)\, a_3 \quad (4\text{-}37)$$

Checking to see if the weighting factor ratios (4-36) and (4-37) minimize or maximize the objective functional, we used the second derivative test. We have

$$\frac{\partial^2 \phi}{\partial (c_1/c_3)^2} = \frac{2.5816}{a_3} , \quad \frac{\partial^2 \phi}{\partial (c_2/c_3)^2} = \frac{0.04998}{a_3} , \text{ and}$$

$$\frac{\partial^2 \phi}{\partial (c_1/c_3) \partial (c_2/c_3)} = \frac{0.3159}{a_3}$$

Since $\dfrac{\partial^2 \phi}{\partial (c_1/c_3)^2} > 0$ and

$$\frac{\partial^2 \phi}{\partial (c_1/c_3)^2} \frac{\partial^2 \phi}{\partial (c_2/c_3)^2} - \left[\frac{\partial \phi}{\partial (c_1/c_3) \partial (c_2/c_3)} \right]^2 = \frac{0.02924}{a_3^{\ 2}} > 0$$

The objective functional is minimized for this particular set of weighting factors.

Substituting the optimum values of the weighting factor ratios into the expressions for the performance indices and the objective functionals, we get their minimum values in terms of the initial conditions, x_{10} and x_{20}, and the BOD level criterion, u_m, for the dumping of organic waste.

$$z_1 = 0.4469 \ x_{10} + 0.0000 \ x_{20} + 1.3738 \ u_m \tag{4-38}$$

$$z_2 = 0.1021 \ x_{10} + 0.1280 \ x_{20} + 0.1784 \ u_m \tag{4-39}$$

$$z_3 = a_3 \ [0.1034 \ x_{10}^{\ 2} + 0.5427 \ x_{20}^{\ 2} + 0.2776 \ u_m^{\ 2}$$
$$+ 0.4018 \ x_{10} \ x_{20} + 0.2106 \ x_{10} \ u_m + 0.08724 \ x_{20} \ u_m] \tag{4-40}$$

$$\text{Min } \phi = -a_3[0.3101\ x_{10}^2 + 1.6276\ x_{20}^2 + 0.8329\ u_m^2$$

$$+ 1.2051\ x_{10}\ x_{20} + 0.6320\ x_{10}\ u_m$$

$$+ 0.2620\ x_{20}\ u_m] \tag{4-41}$$

After carrying out the appropriate calculations, we are able to establish the following relationship:

$$a_3 \begin{bmatrix} z_1 \\ \\ z_2 \end{bmatrix} = \begin{bmatrix} 1.7170 & 0.2103 \\ \\ 0.2104 & 0.03329 \end{bmatrix} \begin{bmatrix} (c_1/c_3) \\ \\ (c_2/c_3) \end{bmatrix} \tag{4-42}$$

To determine the shape of u_1, x_1 and x_2 for realistic values of x_{10}, x_{20} and u_m, we use the same values as Perlis and Cook [18], namely x_{10} = 30 mg/ℓ, x_{20} = 3.06 mg/ℓ, and u_m = 23 mg/ℓ, to evaluate the system of equations. The results for the dumping control, the level of BOD and the saturation level of DO minus the actual level of DO are plotted in figure 10. The particular values taken on by the weighting factor ratios, the optimum control, the performance indices and the minimum objective functional are furnished below.

$$(c_1/c_3) = -7.0817\ a_3$$

$$(c_2/c_3) = 271.5571\ a_3$$

$$u_1 = 23. - 39.9083 \exp[.16(y-2.5)] + 43.4491 \exp[.66(y-2.5)]$$

$$z_1 = 45.0016$$

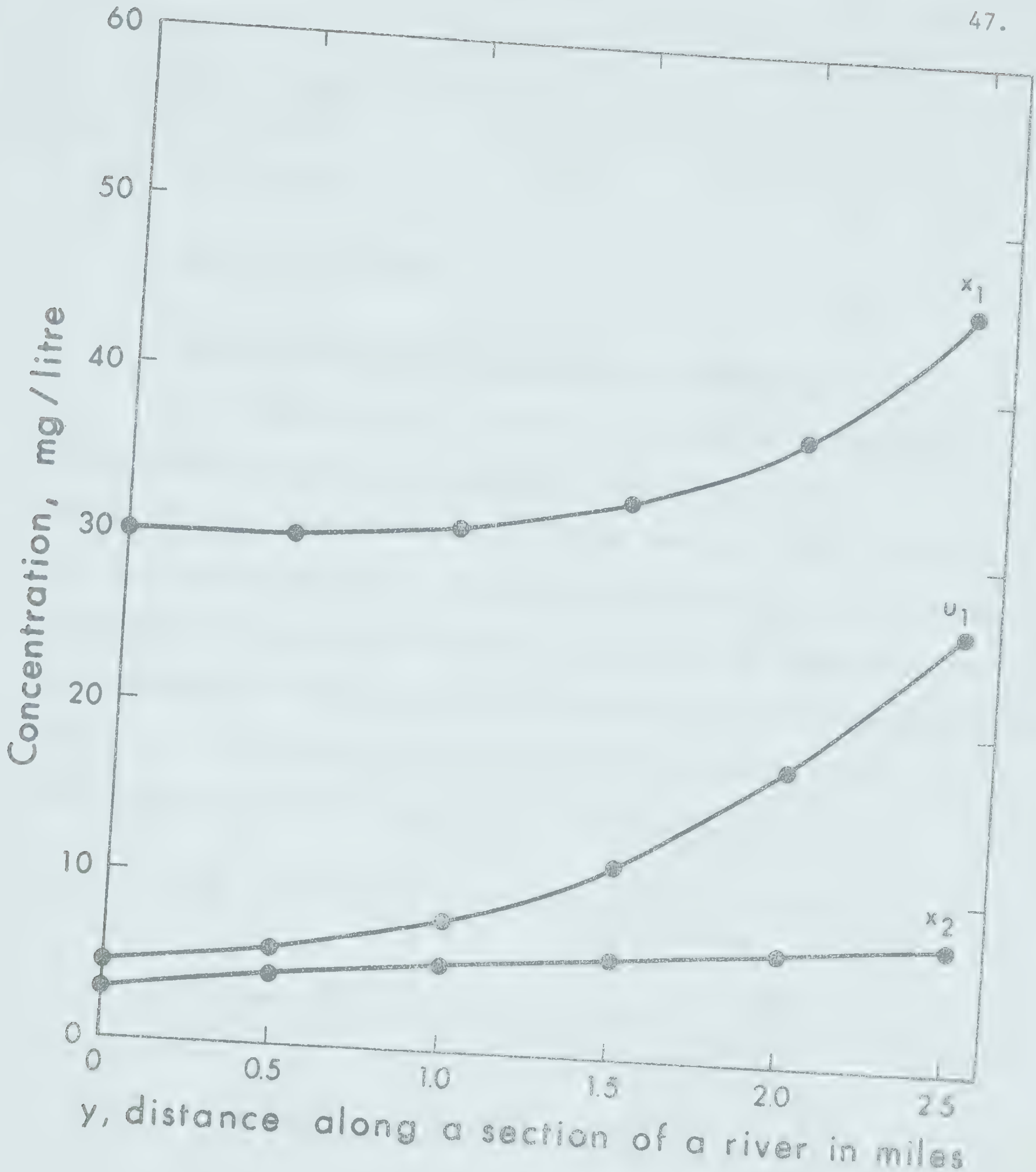

FIGURE 10: Optimum Profiles of u_1, x_1 and x_2 for $\phi = z_3 - (c_1/c_3)z_1 - (c_2/c_3)z_2$

$z_2 = 7.5569$

$z_3 = 433.3604\ a_3$

Min $\phi = -1300.0767\ a_3$

Minimization of the Second Objective Functional

In a similar manner, we proceed to determine the optimum dumping control for the second objective functional, $\phi = a_3\, z_1 \cdot z_3 + z_3$, given in (4-15). As outlined in a previous section, there is no need to solve this problem from the beginning. Since the objective functional is made up of the same cost scales z_1, z_2 and z_3, the solution to the linear combination functional earlier can be used for the second objective functional. Substituting the performance indices (4-32), (4-33) and (4-34) into the new relationship of ϕ, we can write,

$$\phi = \frac{1}{a_3}\left[0.5209\left(\frac{c_1}{c_3}\right)^2 + 0.1307\left(\frac{c_1}{c_3}\right)\left(\frac{c_2}{c_3}\right) + 0.01008\left(\frac{c_2}{c_3}\right)^2\right]$$

$$\left[0.2023\ x_{10} + 0.1652\ x_{20} + 0.4473\ u_m\right]\left(\frac{c_1}{c_3}\right)$$

$$-\left[0.02729\ x_{10} + 0.02022\ x_{20} + 0.06251\ u_m\right]\left(\frac{c_2}{c_3}\right)$$

$$+\ a_3\ [0.1026\ x_{10}^2 + 0.5514\ u_m^2 + 0.1287\ x_{10}\ x_{20}$$

$$+\ 0.4949\ x_{10}\ u_m + 0.3956\ x_{20}\ u_m] \qquad (4\text{-}43)$$

As before the gradient of ϕ is determined and set equal to zero and the ensuing equations are solved to give the following optimum values.

$$(c_1/c_3) = (0.1306\ x_{10} + 0.1757\ x_{20} + 0.2159\ u_m)\ a_3 \qquad (4\text{-}44)$$

$$(c_2/c_3) = (0.5065\ x_{10} - 0.1364\ x_{20} + 1.7009\ u_m)\ a_3 \qquad (4\text{-}45)$$

Using the second derivative test, we can establish whether (4-44) and (4-45) minimize the objective functional (4-43),

$$\frac{\partial^2\phi}{\partial(c_1/c_3)^2} = \frac{1.0418}{a_3}, \quad \frac{\partial^2\phi}{\partial(c_2/c_3)^2} = \frac{0.02016}{a_3}, \text{ and}$$

$$\frac{\partial^2\phi}{\partial(c_1/c_3)\partial(c_2/c_3)} = \frac{0.1307}{a_3}$$

Thus $\quad \dfrac{\partial^2\phi}{\partial(c_1/c_3)^2} > 0$ and

$$\frac{\partial^2\phi}{\partial(c_1/c_3)^2}\ \frac{\partial^2\phi}{\partial(c_2/c_3)^2} - \left[\frac{\partial^2\phi}{\partial(c_1/c_3)\ (c_2/c_3)}\right]^2 = \frac{0.00391}{a_3^2} > 0$$

which fulfills the conditions for the objective functional to be a minimum for this set of the weighting factor ratios. We can evaluate the relationships of the performance indices and the objective functional to determine what their minimum values are for this case.

$$z_1 = 0.5046\ x_{10} - 0.1368\ x_{20} + 1.6957\ u_m \tag{4-46}$$

$$z_2 = 0.1309\ x_{10} + 0.1758\ x_{20} + 0.2165\ u_m \tag{4-47}$$

$$\begin{aligned} z_3 = a_3[&0.01645\ x_{10}^2 + 0.01092\ x_{20}^2 + 0.08281\ u_m^2 \\ &+ 0.02610\ x_{10}\ x_{20} + 0.07353\ x_{10}\ u_m \\ &+ 0.05715\ x_{20}\ u_m] \end{aligned} \tag{4-48}$$

$$\begin{aligned} \text{Min}\quad \phi = a_3[&0.08250\ x_{10}^2 - 0.01313\ x_{20}^2 + 0.4500\ u_m^2 \\ &+ 0.09687\ x_{10}\ x_{20} + 0.4048\ x_{10}\ u_m \\ &+ 0.3256\ x_{20}\ u_m] \end{aligned} \tag{4-49}$$

Using the same values for x_{10}, x_{20} and u_m, the optimum profiles are plotted in figure 11. The values of the optimum weighting factor ratios, the optimum control, the performance indices, and the minimum objective functional are

$$(c_1/c_3) = 9.4218\ a_3$$

$$(c_2/c_3) = 53.8978\ a_3$$

$$u_1 = 23. - 13.3346 \exp[.16(y-2.5)] + 8.6237 \exp[.66(y-2.5)]$$

$$z_1 = 53.7198$$

$$z_2 = 9.4453$$

$$z_3 = 115.8698\ a_3$$

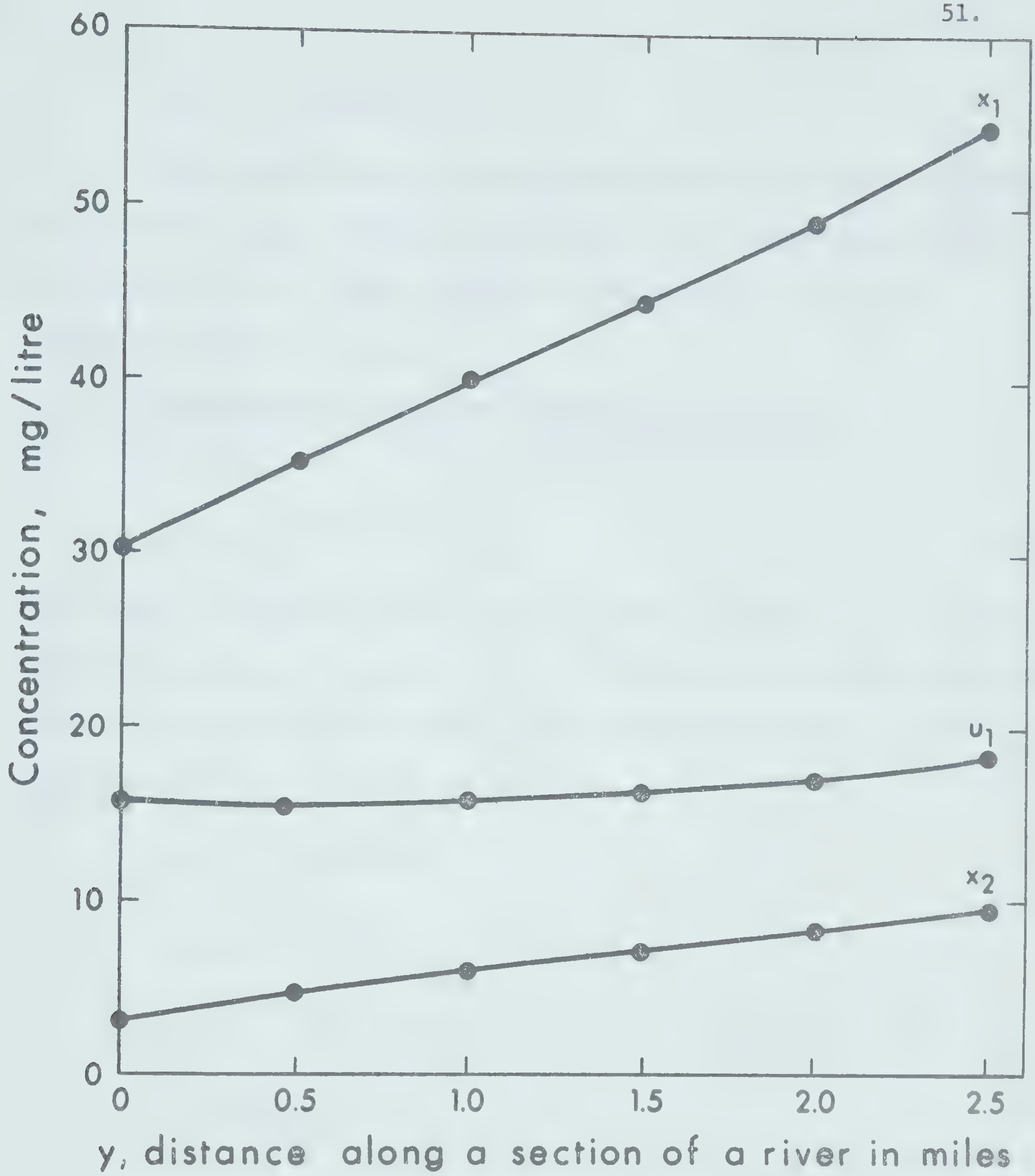

FIGURE 11: Optimum Profiles of u_1, x_1 and x_2 for $\phi = a_3 z_1 \cdot z_2 + z_3$

Min $\phi = 623.2671\ a_3$

From these results we can conclude that the optimum weighting factor ratio, c_1/c_3, is approximately equal to the performance index, z_2, multiplied by a_3, while the other optimum ratio, c_2/c_3, is approximately equal to $a_3 z_1$.

Minimization of the Third Objective Functional

We have

$$\phi = a_3(z_1^2 + z_2^2)/2 + z_3 \tag{4-50}$$

The optimum profiles for this case are plotted in figure 12. The same values of x_{10}, x_{20} and u_m are used. The values for the optimum weighting factor ratios, the optimum control, the performance indices and the objective functional turn out to be

$(c_1/c_3) = 35.8597\ a_3$

$(c_2/c_3) = 7.4586\ a_3$

$u_1 = 23. - 19.1232 \exp[.16(y-2.5)] + 1.1934 \exp[.66(y-2.5)]$

$z_1 = 35.8601$

$z_2 = 7.4350$

$z_3 = 581.9590\ a_3$

Min $\phi = 1252.5713\ a_3$

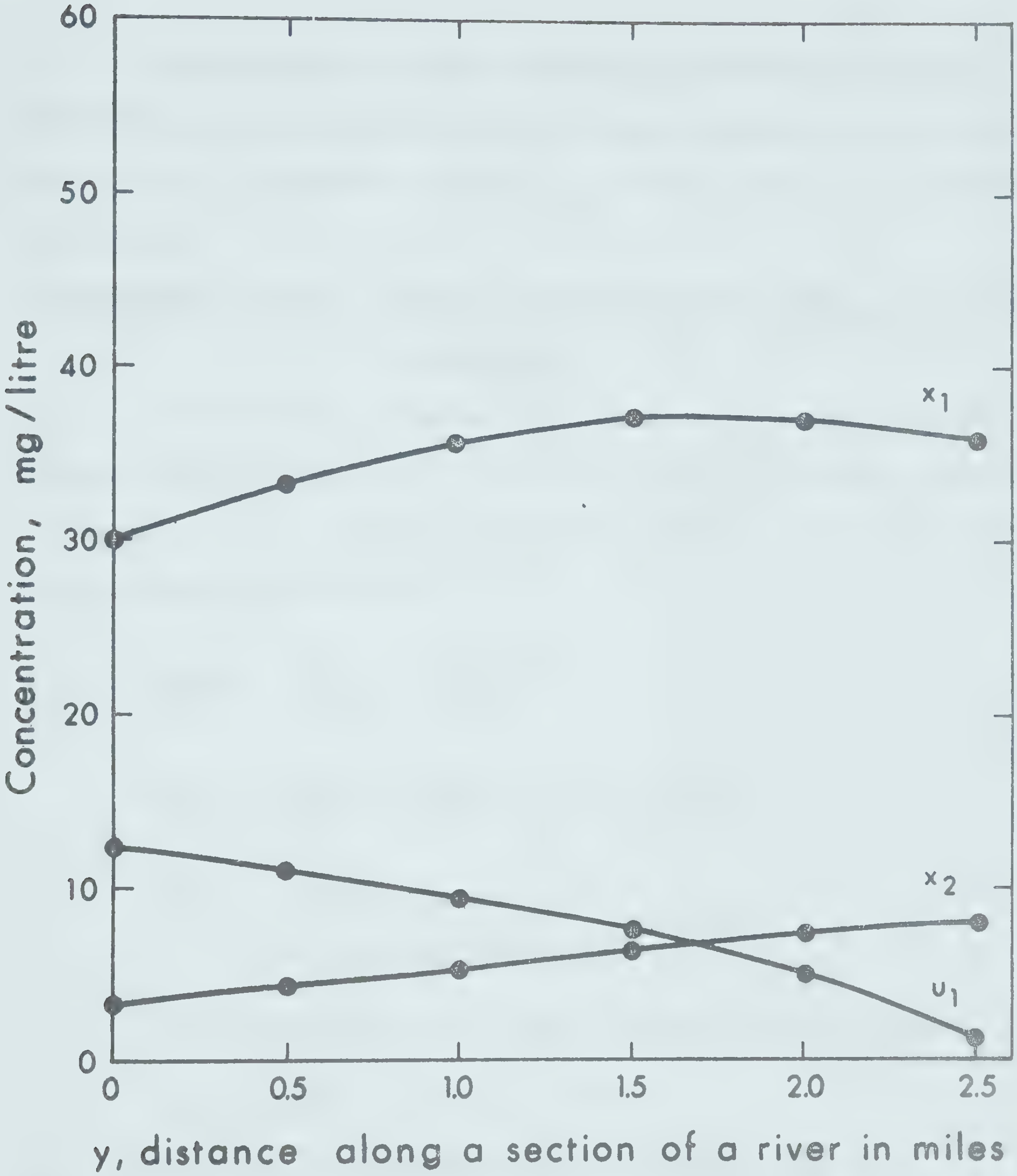

FIGURE 12: Optimum Profiles of u_1, x_1 and x_2 for $\phi = a_3(z_1^2 + z_2^2)/2 + z_3$

Again we notice a direct relationship between the optimum weighting factor ratios and the two performance indices to state that the ratio c_1/c_3 approximately equals a_3z_1 and the ratio c_2/c_3 approximately equals a_3z_2.

4.4 Management of Water Quality by Controlled Dumping (BOD) and Artificial Aeration (DO)

In this section, let us consider adding an aeration control to the same section of the river to see what effect it has on the profiles BOD and (DO_{SAT} - DO). Now the differential equations, (2-7) and (2-8), for the system are written as

$$\left.\begin{aligned} \frac{dB(y)}{dy} &= -\frac{K_r}{V}B(y) + \frac{u_1'(y)}{V} \\ \frac{dD(y)}{dy} &= -\frac{K_d}{V}B(y) + \frac{K_a}{V}[D_s - D(y)] + \frac{u_2'(y)}{V} \end{aligned}\right\} \qquad (4\text{-}51)$$

where u_1' is the dumping control

u_2' is the aeration control

As in the previous case, the state variables are given by

$$x_1(y) = B(y), \quad x_2(y) = D_s - D(y)$$

which gives the above set of equations in matrix form as

$$\begin{bmatrix} \dot{x}_1(y) \\ \dot{x}_2(y) \end{bmatrix} = \begin{bmatrix} -B & 0 \\ C & -A \end{bmatrix} \begin{bmatrix} x_1(y) \\ x_2(y) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1(y) \\ u_2(y) \end{bmatrix} \qquad (4\text{-}52)$$

This system is also state controllable.

The performance indices, J_1 and J_2, will remain the same but J_3 will be changed to accommodate the second control, u_2.

$$J_3 = \int_0^{y_f} \left\{ a_1 + [-a_2 0] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + [u_1 \ u_2] \begin{bmatrix} a_3 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\} dy \qquad (4\text{-}53)$$

We consider first the linear combination of these indices as objective functional.

Solution to the Linear Combination Problem

The augmented system of equations is given by

$$\left.\begin{aligned}
\dot{x}_1(y) &= -B\, x_1(y) + u_1(y) & x_1(0) &= x_{10} \\
\dot{x}_2(y) &= C\, x_1(y) - A\, x_2(y) - u_2(y) & x_2(0) &= x_{20} \\
\dot{z}_1(y) &= 0 & z_1(0) &= x_{10} \\
\dot{z}_2(y) &= 0 & z_2(0) &= x_{20} \\
\dot{z}_3(y) &= a_1 - a_2\, u_1 + a_3\, u_1^2 + b\, u_2^2 & z_3(0) &= 0
\end{aligned}\right\} \qquad (4\text{-}54)$$

The objective functional is

$$\phi = c_1\, z_1 + c_2\, z_2 + c_3\, z_3$$

The Hamiltonian

$$H = c_3(a_1 - a_2\, u_1 + a_3\, u_1^2 + bu_2) - B\, x_1\, p_1 + u_1\, p_1 +$$

$$+ C\, x_1\, p_2 - A\, x_2\, p_2 - u_2\, p_2 \qquad (4\text{-}55)$$

The costate equations and the solutions to them are the same as before. There are no constraints on either control, hence the optimum dumping control is

$$u_1(y) = u_m - K_1\, e^{B(y-y_f)} - K_2\, e^{A(y-y_f)} \tag{4-56}$$

where
$$K_1 = \frac{1}{2a_3}\left[\left(\frac{c_1}{c_3}\right) - \frac{C}{(B-A)}\left(\frac{c_2}{c_3}\right)\right]$$

$$K_2 = \frac{1}{2a_3}\left[\frac{C}{(B-A)}\left(\frac{c_2}{c_3}\right)\right]$$

and the optimum aeration control is

$$u_2(y) = \frac{e^{A(y-y_f)}}{2b}\left(\frac{c_2}{c_3}\right) \tag{4-57}$$

Substituting the optimum controls back into the system of equations (4-54), the optimum BOD profile is

$$x_1(y) = x_{10}\, e^{-By} + u_m \frac{(1-e^{-By})}{B} - K_1 \frac{(e^{B(y-y_f)} - e^{-B(y-y_f)})}{2B}$$

$$- K_2 \frac{(e^{A(y-y_f)} - e^{-By-Ay_f})}{(B+A)} \tag{4-58}$$

and the optimum $DO_{SAT} - DO(y)$ profile is

$$x_2(y) = x_{20}\, e^{-Ay} + \frac{C}{(B-A)}\left[x_{10}\, e^{-Ay} + u_m \frac{(1-e^{-Ay})}{A}\right.$$

$$\left. - K_1 \frac{(e^{B(y-y_f)} - e^{-Ay-By_f})}{(B+A)} - K_2 \frac{(e^{A(y-y_f)} - e^{-A(y+y_f)})}{2A} - x_1(y)\right]$$

$$- \frac{1}{2b_1} \frac{(e^{A(y-y_f)} - e^{-A(y+y_f)})}{2A}\left(\frac{c_2}{c_3}\right) \tag{4-59}$$

The new performance indices become

$$z_1 = x_{10}\, e^{-By_f} + u_m \frac{(1-e^{-By_f})}{B} - K_1 \frac{(1-e^{-2By_f})}{2B} - K_2 \frac{(1-e^{-(B+A)y_f})}{(B+A)} \qquad (4\text{-}60)$$

$$z_2 = x_{20}\, e^{-Ay_f} + \frac{C}{(B-A)} \left[x_{10}\, e^{-Ay_f} + u_m \frac{(1-e^{-Ay_f})}{A} - K_1 \frac{(1-e^{-(B+A)y_f})}{(B+A)} - K_2 \frac{(1-e^{-2Ay_f})}{2A} - z_1 \right] - \frac{1}{2b_1} \frac{(1-e^{-2Ay_f})}{2A} \left(\frac{c_2}{c_3}\right) \qquad (4\text{-}61)$$

and

$$z_3 = a_3 \left[K_1^2 \frac{(1-e^{-2By_f})}{2B} + 2K_1K_2 \frac{(1-e^{-(B+A)y_f})}{(B+A)} + K_2^2 \frac{(1-e^{-2Ay_f})}{2A} + \frac{1}{4b_1} \frac{(1-e^{-2Ay_f})}{2A} \left(\frac{c_2}{c_3}\right)^2 \right] \qquad (4\text{-}62)$$

Using the same values as given by (4-28), the three performance indices are

$$z_1 = 0.6703\, x_{10} + 2.0606\, u_m - \frac{1}{a_3} \left[0.8605 \left(\frac{c_1}{c_3}\right) + 0.1053 \left(\frac{c_2}{c_3}\right) \right] \qquad (4\text{-}63)$$

$$z_2 = 0.1531\, x_{10} + 0.1920\, x_{20} + 0.2676\, u_m - \frac{1}{a_3} \left[0.1053 \frac{c_1}{c_3} + \left(0.01666 + 0.3648 \frac{a_3}{b} \right) \left(\frac{c_2}{c_3}\right) \right] \qquad (4\text{-}64)$$

$$z_3 = \frac{1}{a_3}\left[0.4303\left(\frac{c_1}{c_3}\right)^2 + 0.1053\,\frac{c_1}{c_3}\left(\frac{c_2}{c_3}\right)\right.$$

$$\left. + \left[0.00833 + 0.1824\,\frac{a_3}{b}\right]\left(\frac{c_2}{c_3}\right)^2\right] \tag{4-65}$$

Having solved the linear combination problem, we now optimize the system with respect to the three objective functionals as given by the equations (4-63) thru (4-65).

Minimization of the First Objective Functional

We have

$$\phi = z_3 - (c_1/c_3)\, z_1 - (c_2/c_3)\, z_2$$

$$= \frac{1}{a_3}\left\{1.2908\left(\frac{c_1}{c_3}\right)^2 + 0.3159\left(\frac{c_1}{c_3}\right)\left(\frac{c_2}{c_3}\right) + \left[0.02499 + 0.5472\,\frac{a_3}{b}\right]\left(\frac{c_2}{c_3}\right)^2\right.$$

$$- \left[0.6703\, x_{10} + 2.0606\, u_m\right]\left(\frac{c_1}{c_3}\right) - \left[0.1531\, x_{10}\right.$$

$$\left.\left. + 0.1920\, x_{20} + 0.2676\, u_m\right]\left(\frac{c_2}{c_3}\right)\right\} \tag{4-66}$$

The optimum weighting factors are determined in the same manner as before to give

$$\left(\frac{c_1}{c_3}\right) = \frac{[(-0.01822+0.8995a_3/b)x_{10}-0.07438x_{20}+(0.02263+2.7652a_3/b)u_m]a_3}{(0.03585 + 3.4644\, a_3/b)} \tag{4-67}$$

$$\left(\frac{c_2}{c_3}\right) = \frac{[0.2250x_{10}+0.6078x_{20}+0.04891u_m]a_3}{(0.03585 + 3.4644\, a_3/b)} \tag{4-68}$$

To demonstrate that this minimizes the objective functional we use the second derivative test,

$$\frac{\partial^2 \phi}{\partial (c_1/c_3)^2} = \frac{2.5816}{a_3} , \quad \frac{\partial^2 \phi}{\partial (c_1/c_3) \partial (c_2/c_3)} = \frac{0.3159}{a_3} , \quad \text{and}$$

$$\frac{\partial^2 \phi}{\partial (c_2/c_3)^2} = \frac{(0.04998 + 1.0944 \ a_3/b)}{a_3}$$

Then $\quad \dfrac{\partial^2 \phi}{\partial (c_1/c_3)^2} > 0$ and

$$\frac{\partial^2 \phi}{\partial (c_1/c_3)^2} \ \frac{\partial^2 \phi}{\partial (c_2/c_3)^2} - \left[\frac{\partial \phi}{\partial (c_1/c_3) \partial (c_2/c_3)} \right]^2$$

$$= \frac{(0.02924 + 2.8530 \ a_3/b)}{a_3^{\ 2}} > 0$$

for $a_3/b > - 0.01025$.

To illustrate the optimum profiles (see figure 13) for this objective function we choose the ratio of a_3/b to be equal to unity. The other parameters remain the same as in previous examples. The optimum weighting factors, the resulting expression for the performance indices and the objective functional are

$$\left(\frac{c_1}{c_3}\right) = [0.2518 \ x_{10} - 0.02125 \ x_{20} + 0.7965 \ u_m] \ a_3 \tag{4-69}$$

$$\left(\frac{c_2}{c_3}\right) = [0.06427 \ x_{10} + 0.1737 \ x_{20} + 0.01395 \ u_m] \ a_3 \tag{4-70}$$

$$z_1 = 0.4469 \ x_{10} + 0.0000 \ x_{20} + 1.3738 \ u_m \tag{4-71}$$

$$z_1 = 0.1021 \ x_{10} + 0.1280 \ x_{20} + 0.1784 \ u_m \tag{4-72}$$

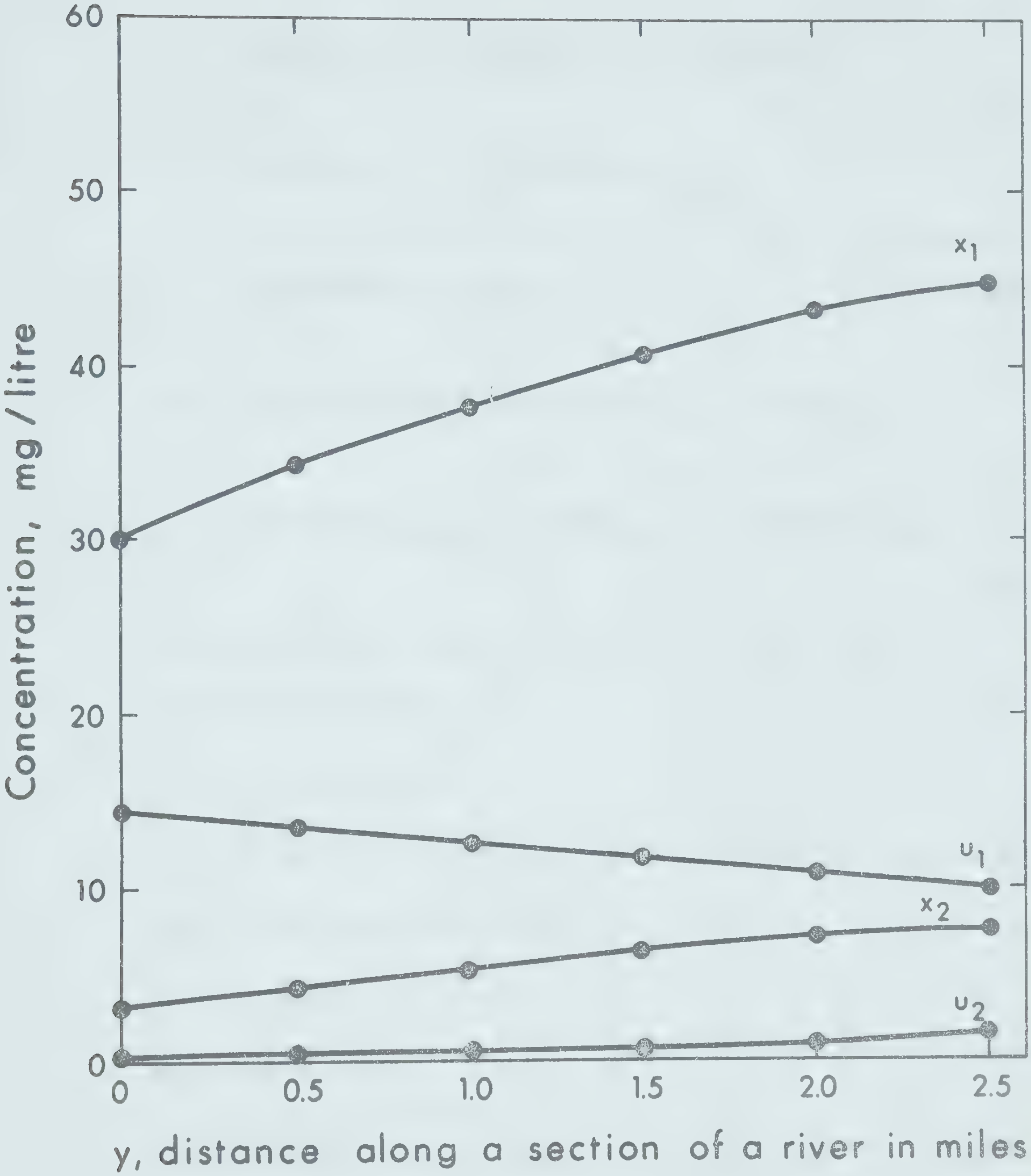

FIGURE 13: Optimum Profiles of u_1, u_2, x_1 and x_2 for $\phi = z_3 - (c_1/c_3)z_1 - (c_2/c_3)z_2$

$$z_3 = [0.02977\ x_{10}^2 + 0.005558\ x_{20}^2 + 0.2742\ u_m^2$$

$$+ 0.004113\ x_{10}\ x_{20} + 0.1787\ x_{10}\ u_m$$

$$+ 0.0008919\ x_{20}\ u_m]\ a_3 \qquad (4\text{-}73)$$

$$\text{Min}\ \ \phi = -[0.08930\ x_{10}^2 + 0.01667\ x_{20}^2 + 0.8225\ u_m^2$$

$$+ 0.01234\ x_{10}\ x_{20} + 0.5360\ x_{10}\ u_m + 0.002684\ x_{20}\ u_m]\ a_3 \qquad (4\text{-}74)$$

For $x_{10} = 30$ mg/ℓ, $x_{20} = 3.06$ mg/ℓ and $u_m = 23$ mg/ℓ

$(c_1/c_3) = 25.8070\ a_3$

$(c_2/c_3) = 2.7812\ a_3$

$u_1 = 23. - 13.3485 \exp [.16(y-2.5)] + 0.4450 \exp [.66(y-2.5)]$

$u_2 = 1.3906 \exp [.66(y-2.5)]$

$z_1 = 45.0030$

$z_2 = 7.5569$

$z_3 = 295.6123\ a_3$

$\text{Min}\ \phi = -886.7966\ a_3$

We notice that two of the performance indices are related to the weighting factor ratios by

$$a_3\begin{bmatrix} z_1 \\ \\ z_2 \end{bmatrix} = \begin{bmatrix} 1.7301 & 0.1721 \\ & \\ 0.2108 & 0.7626 \end{bmatrix} \begin{bmatrix} c_1/c_3 \\ \\ c_2/c_3 \end{bmatrix} \qquad (4\text{-}75)$$

To compare the optimum solutions of the state variables for the system with the aeration control and without this control, the optimum trajectories of the state vector, $[x_1\ x_2]^T$, are shown for the section of the river under consideration in the figure 14 on the $x_1\ x_2$ - plane. From the plot for the state plane, we find the initial and final conditions of the trajectories are the same for both problems when the optimum controls are applied to the system.

Minimization of the Second Objective Functional

The second objective functional, $\phi = a_3 z_1 \cdot z_2 + z_3$, for the case of the aeration control takes on minimum value when

$$(c_1/c_3) = [-0.09677\ x_{10} + 0.2375\ x_{20} - 0.5486\ u_m]\ a_3 \qquad (4\text{-}76)$$

$$(c_2/c_3) = [0.6818\ x_{10} - 0.1848\ x_{20} + 2.2915\ u_m]\ a_3 \qquad (4\text{-}77)$$

For $x_{10} = 30$ mg/ℓ, $x_{20} = 3.06$ mg/ℓ and $u_m = 23$ mg/ℓ, we plot the optimum dumping control, the optimum aeration control and the resulting state variable profiles in figure 15. In addition the weighting factor ratios, optimum controls, performance indices and minimum objective functional becomes

$$(c_1/c_3) = -14.7932\ a_3$$

$$(c_2/c_3) = 72.5909\ a_3$$

$$u_1 = 23. - 4.2179 \exp[.16(y-2.5)] + 11.6145 \exp[.66(y-2.5)]$$

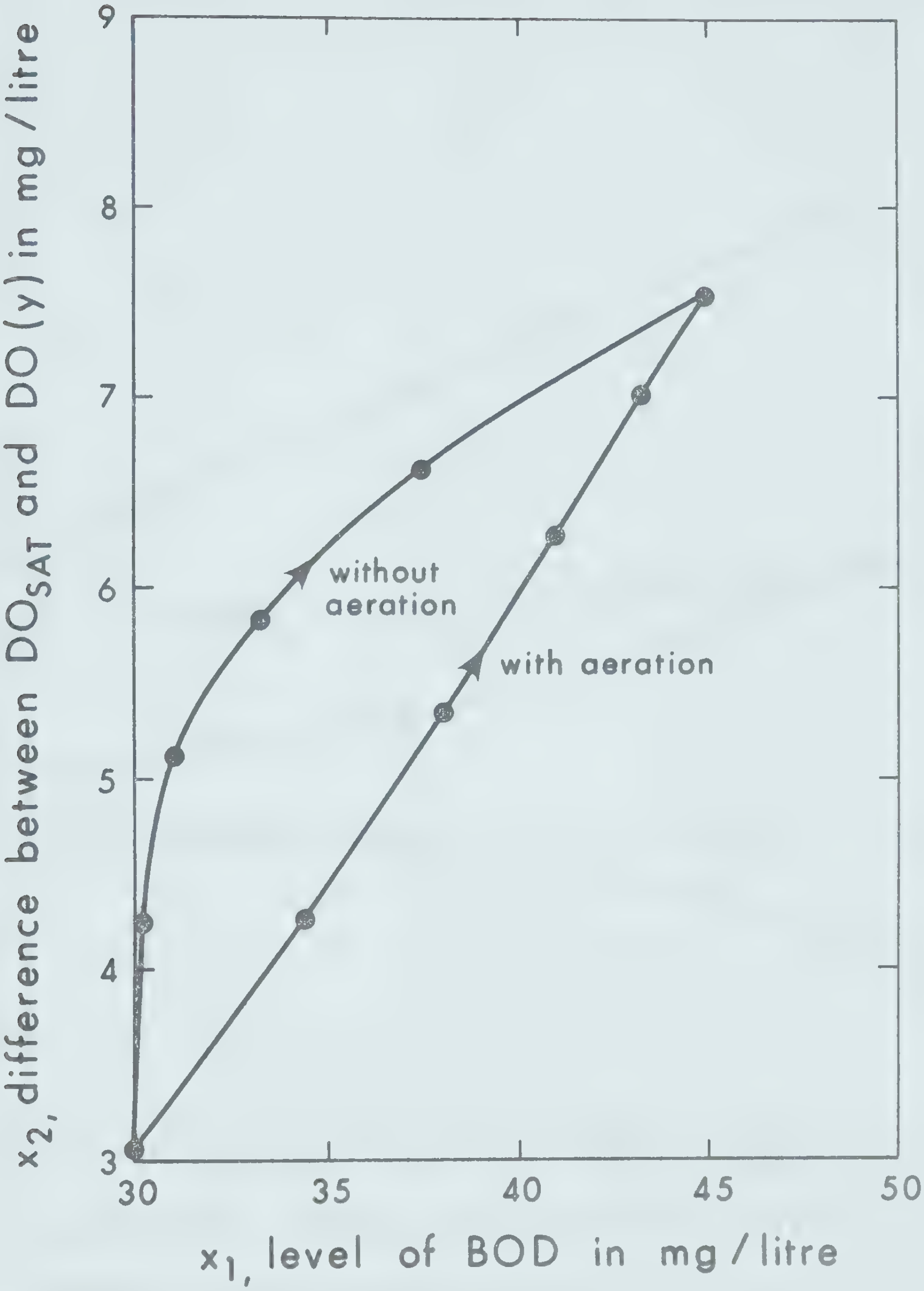

FIGURE 14: State Plane for $\phi = z_3 - (c_1/c_3)z_1 - (c_2/c_3)z_2$

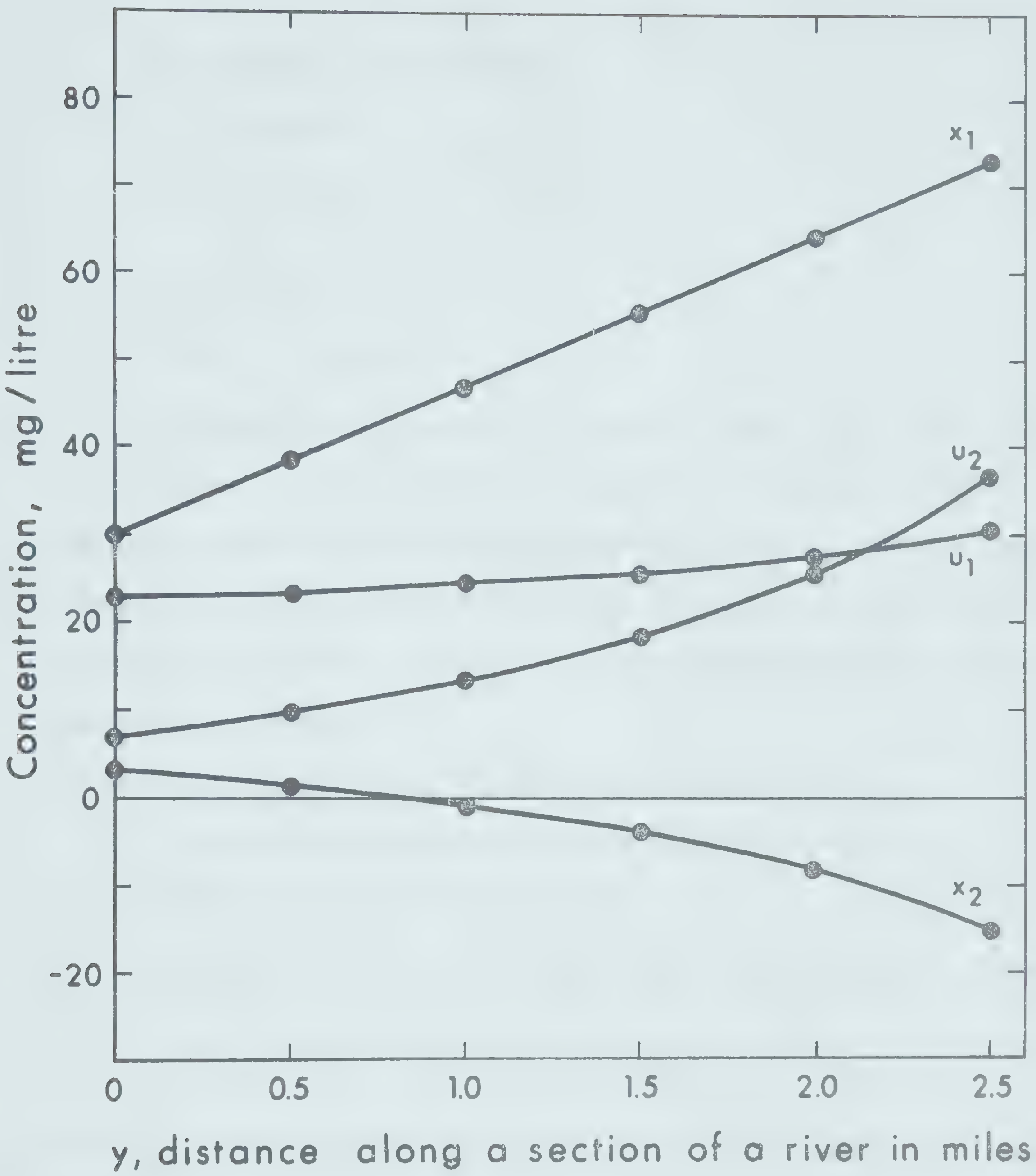

FIGURE 15: Optimum Profiles of u_1, u_2, x_1 and x_2 for $\phi = a_3 z_1 \cdot z_2 + z_3$

$u_2 = 36.2955 \exp [.66(y-2.5)]$

$z_1 = 72.5885$

$z_2 = -14.7975$

$z_3 = 986.1277\ a_3$

Min $\phi = -87.9966\ a_3$

The relationship between the optimum ratios, c_1/c_3 and c_2/c_3, and the indices, z_1 and z_2, remain the same as the previous problem without the aeration control, but the actual numerical values are not the same. In figure 16, the state plane trajectories are plotted for the system with and without the aeration control. It is to be noted that the trajectories diverge from one another.

Minimization of the Third Objective Functional

For the third objective functional, $\phi = a_3(z_1^2 + z_2^2)/2 + z_3$, we get the optimum weighting factor ratios as,

$$(c_1/c_3) = [0.3555\ x_{10} - 0.007901\ x_{20} + 1.1013\ u_m]\ a_3 \qquad (4\text{-}78)$$

$$(c_2/c_3) = [0.08372\ x_{10} + 0.1396\ x_{20} + 0.1093\ u_m]\ a_3 \qquad (4\text{-}79)$$

for the a_3/b equal to unity.

Again we plot the optimum controls and resulting state variables for $x_{10} = 30$ mg/ℓ, $x_{20} = 3.06$ mg/ℓ and $u_m = 23$ mg/ℓ (see figure 17), plus evaluate the following,

$(c_1/c_3) = 35.9709\ a_3$

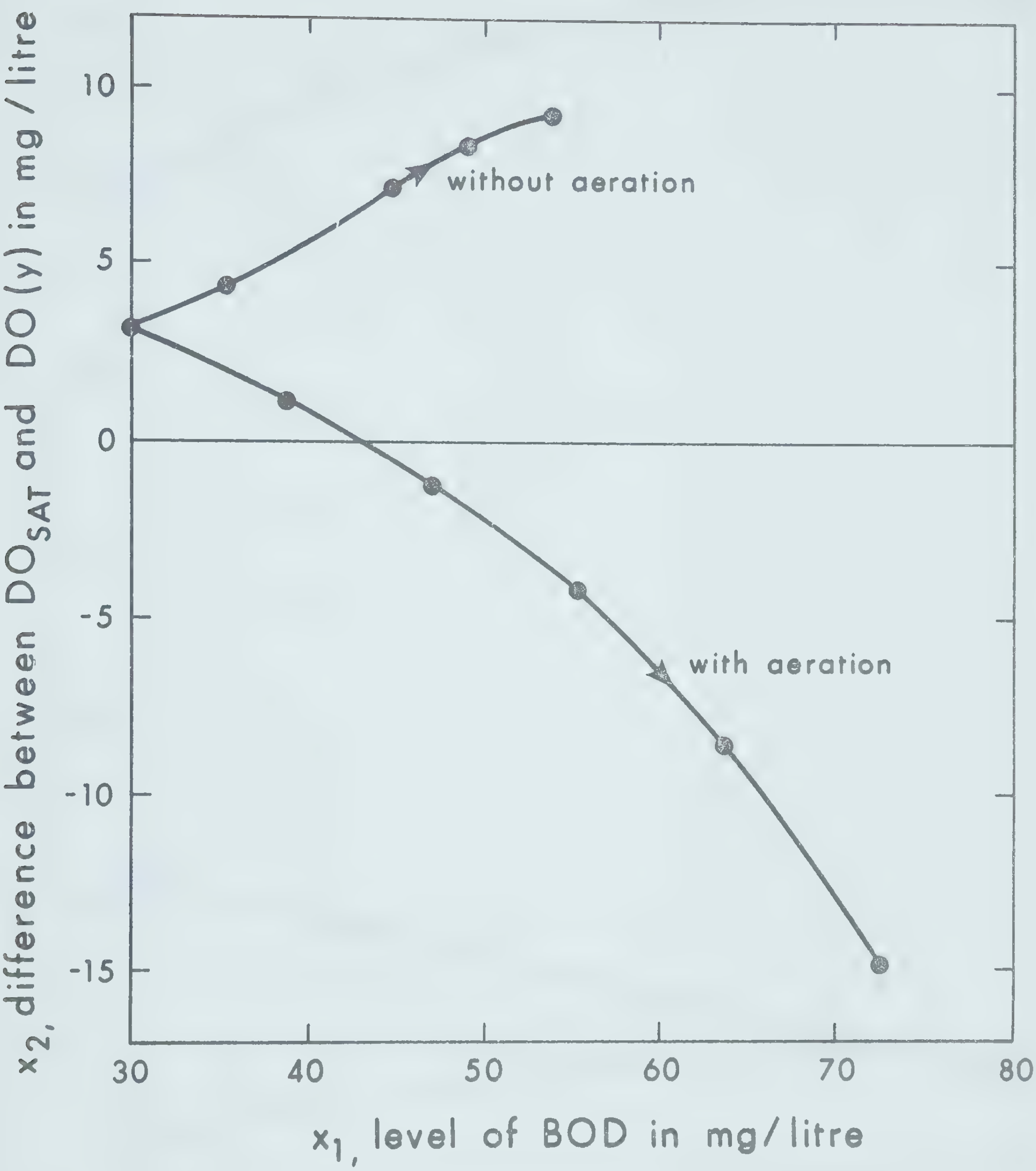

FIGURE 16: State Plane for $\phi = a_3 z_1 \cdot z_2 + z_3$

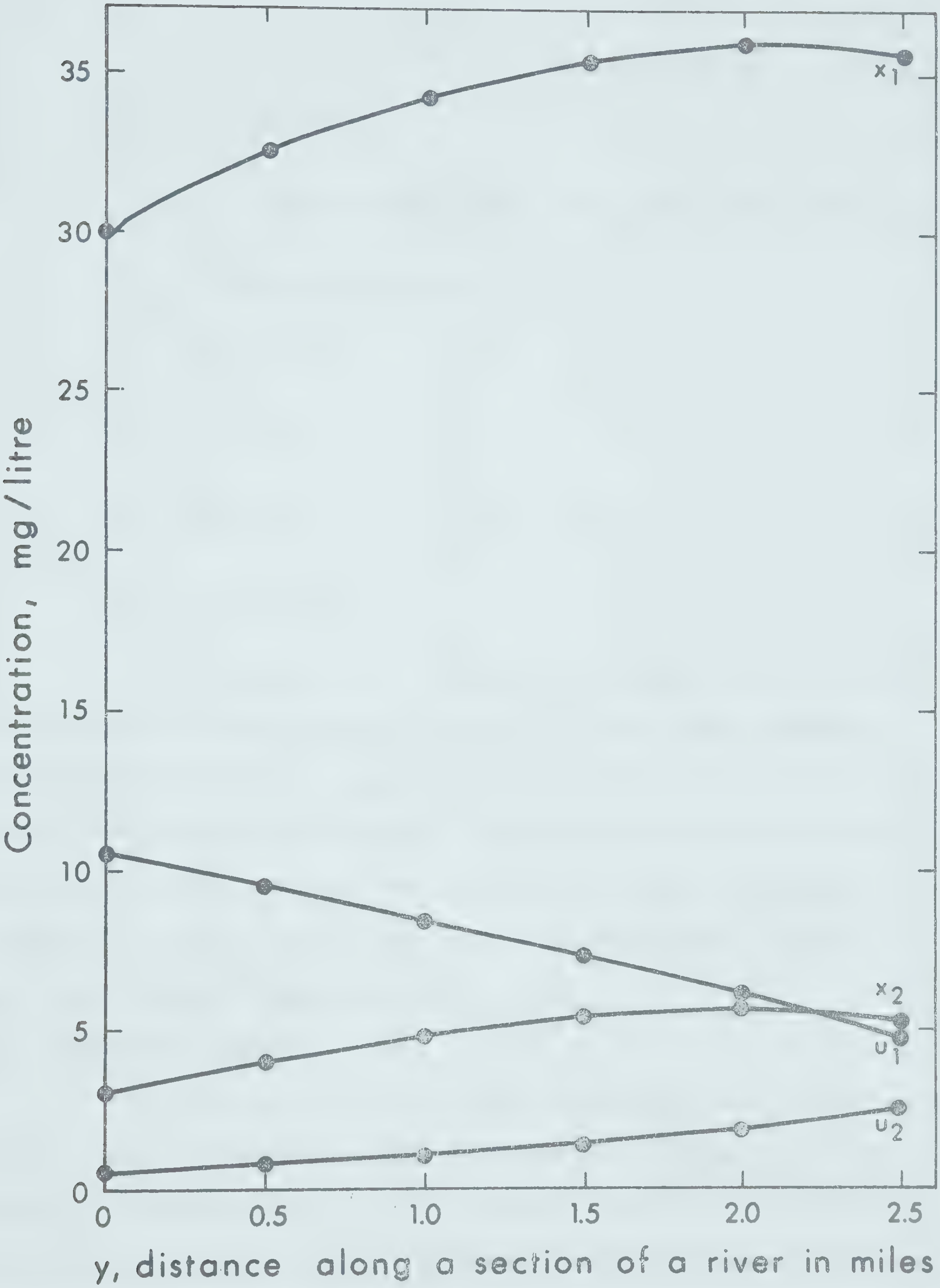

FIGURE 17: Optimum Profiles of u_1, u_2, x_1 and x_2 for $\phi = a_3(z_1^2 + z_2^2)/2 + z_3$

$(c_2/c_3) = 5.4519\ a_3$

$u_1 = 23. - 18.8578 \exp [.16(y-2.5)] + 0.8723 \exp [.66(y-2.5)]$

$u_2 = 2.7260 \exp [.66(y-2.5)]$

$z_1 = 35.9757$

$z_2 = 5.4679$

$z_3 = 583.0876\ a_3$

$\text{Min}\ \ \phi = 1245.1621\ a_3$

As in the case of the system with the dumping control only, we have the optimum weighting factor ratios equal to the performance indices multiplied by a_3. Again plotting the trajectories for the state plane in figure 18, we are able to conclude that the trajectories are similar. In other words one can be considered to be a reduced mirror image of the other and the final conditions of the state variable, x_1, are nearly equal in magnitude for both cases.

4.5 Sensitivity Analysis

In this section we will present a discussion of the sensitivity of the objective functionals used to changes in the weighting factor ratios. Such an analysis yields information about the behaviour of the objective functionals in the neighborhood of their minimum for small deviations in the value of the optimum weighting factor ratios. A quantity called the objective functional sensitivity vector is defined by

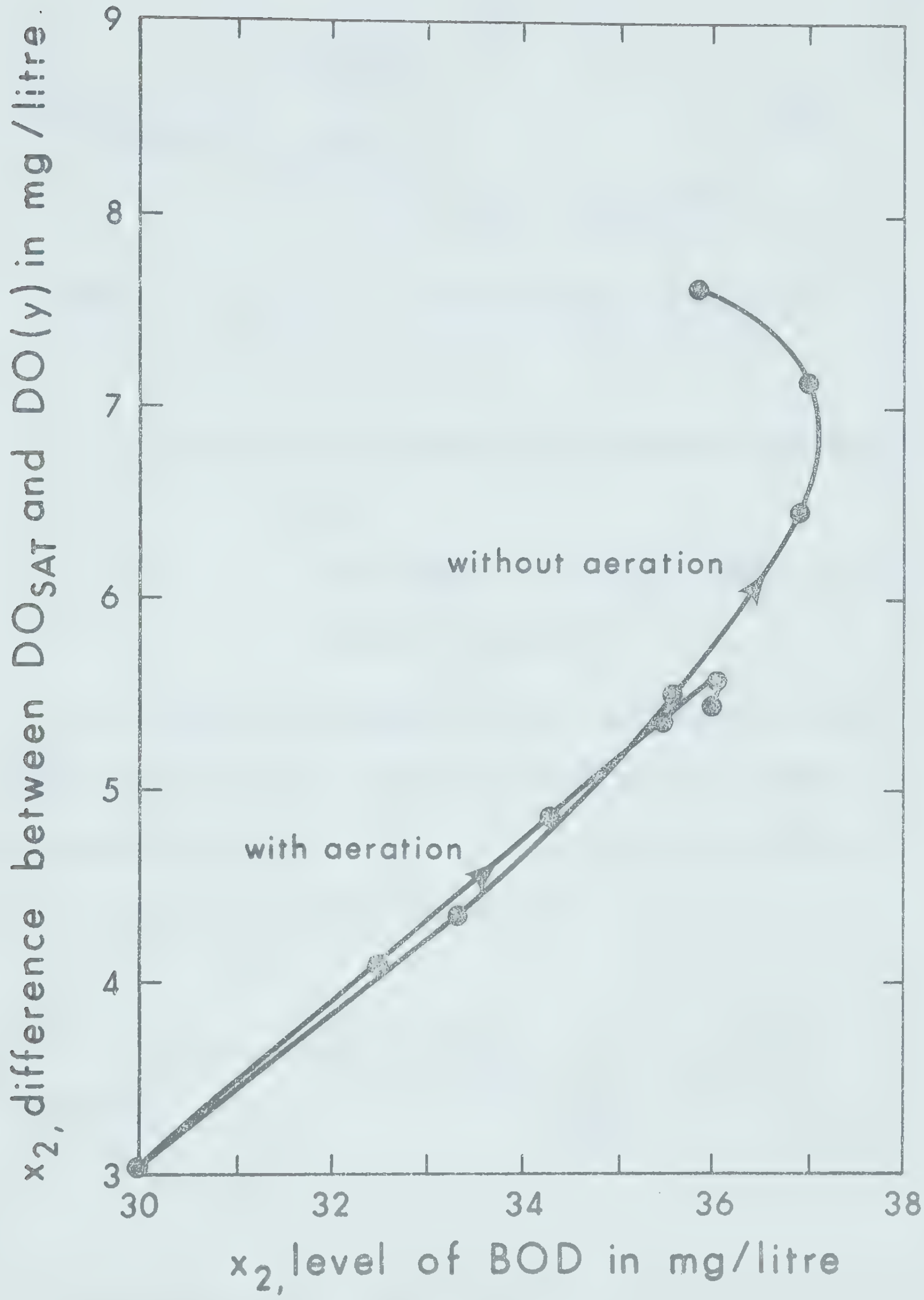

FIGURE 18: State Plane for $a_3(z_1^{\ 2} + z_2^{\ 2})/2 + z_3$

$$\left.\frac{\partial\phi}{\partial\underline{c}}\right|_{\underline{c}=\underline{c}^*+\underline{\Delta}} = \left.\begin{bmatrix} \dfrac{\partial\phi}{\partial(c_1/c_3)} \\ \dfrac{\partial\phi}{\partial(c_2/c_3)} \end{bmatrix}\right|_{c_i/c_3 = (c_i/c_3)^*+\Delta} \quad , \ n=1,2 \qquad (4\text{-}80)$$

where $\underline{c} = \begin{bmatrix} c_1/c_3 \\ c_2/c_3 \end{bmatrix}$ is the weighting factor vector

$(c_i/c_3)^*$ is the value of the optimum weighting factor ratio

Δ is the deviation from the value of the optimum weighting factors

This definition is similar to the performance index sensitivity vector defined by Sage [21]. It is the gradient with respect to the weighting factors evaluated at $\underline{c} = \underline{c}^* + \underline{\Delta}$. Now, the sensitivity of a objective functional can be expressed as

$$S_\phi = \left.\frac{\partial\phi}{\partial\underline{c}}\right|^T_{\underline{c}=\underline{c}^*+\underline{\Delta}} \underline{\Delta} = \Delta\phi = d\phi = \left.\frac{\partial\phi}{\partial(c_1/c_3)}\right|_{\frac{c_i}{c_3}=\frac{c_i^*}{c_3}+\Delta} \Delta + \left.\frac{\partial\phi}{\partial(c_2/c_3)}\right|_{\frac{c_i}{c_3}=\frac{c_i^*}{c_3}+\Delta} \Delta \qquad (4\text{-}81)$$

The values of the sensitivity and sensitivity vector are listed in table 3 for each objective functional.

Objective Functionals	Without the Aeration Control			With the Aeration Control		
	$\partial\phi/\partial(c_1/c_3)$	$\partial\phi/\partial(c_2/c_3$	$S\phi$	$\partial\phi/\partial(c_1/c_3)$	$\partial\phi/\partial(c_2/c_3)$	$S\phi$
$\phi=z_3-\frac{c_1}{c_3}z_1-\frac{c_2}{c_3}z_2$	2.8975Δ	0.3659Δ	$3.2634\Delta^2$	2.8975Δ	1.4603Δ	$4.3578\Delta^2$
$\phi=a_3z_1\cdot z_2+z_3$	1.1715Δ	0.1509Δ	$1.3224\Delta^2$	1.4864Δ	0.9064Δ	$2.3928\Delta^2$
$\phi=a_3\frac{(z_1^2+z_2^2)}{2}+z_3$	1.8099Δ	0.2257Δ	$2.0356\Delta^2$	1.8483Δ	0.7742Δ	$2.6225\Delta^2$

Table 3. Sensitivity Analysis

From the table, it is seen that the first objective functional is most sensitive to the deviation of the weighting factors from their optimum value while the second objective function is the least sensitive in both cases without the aeration control and with the aeration control. In general for all three objective functionals, the sensitivity vectors are most sensitive to the deviation in the weighting factor, c_1/c_3, than the weighting factor, c_2/c_3.

4.6 Interaction of Performance Indices

Next, we shall relate the performance indices, z_1, z_2 and z_3, to each other to investigate how they vary with regards to each other for both cases: without and with aeration. Turning our attention to the equation for the performance indices, (4-32) thru (4-34) and (4-63) thru (4-65), we notice that the three equations are functions of two variables, c_1/c_3 and c_2/c_3. This allows us to combine the equations into single equation while eliminating the weighting factors from it. Using the same parameter as before, the equation for system I becomes

$$z_3/a_3 = 2.5660\ z_1^2 - 32.4373\ z_1 z_2 + 132.5219\ z_2^2 + 21.2575\ z_1 - 814.7469\ z_2 + 3900.2377 \qquad (4\text{-}82)$$

and the equation for system II becomes

$$z_3/a_3 = 0.6014\ z_1^2 - 0.3319\ z_1 z_2 + 1.3560\ z_2^2 - 77.4346\ z_1 - 8.3379\ z_2 + 2660.5918 \qquad (4\text{-}83)$$

Both equations are second degree in z_1, z_2 and z_3. To determine the classification of these quadric surfaces, a rotation and translation or coordinates in the three-dimensional space will be performed. The rotation of coordinates in the plane, $z_1 = u\cos\theta - v\sin\theta$ and $z_2 = u\sin\theta + v\cos\theta$, is used to dispose of the $z_1 z_2$ term in each equation. Rotating the relationships by 7.005° and 11.872° for system I and system II, respectively, the equations of the surface becomes

$$z_3/a_3 = 0.5725\,u^2 + 134.5154\,v^2 - 78.2621\,u - 811.2579\,v + 3900.2377 \tag{4-84}$$

$$z_3/a_3 = 0.5665\,u^2 + 1.3909\,v^2 - 77.4936\,u + 7.7706\,v + 2660.5918 \tag{4-85}$$

By introducing the translation of coordinates, $x = u-68.3496$, $y = v-3.0155$ and $z = z_3/a_3-2.4767$, we find, for the equation of the surface of system I,

$$\frac{x^2}{(1.3216)^2} + \frac{y^2}{(0.08622)^2} = z \tag{4-86}$$

which we recognize as an elliptic paraboloid.

The translation of coordinates, $x = u-68.3968$, $y = v+2.7934$, and $z = z_3/a_3+0.4178$, shows that the surface of the performance indices for system II is also an elliptic paraboloid with the equation

$$\frac{x^2}{(1.3286)^2} + \frac{y^2}{(0.8479)^2} = z \tag{4-87}$$

The two surfaces are shown in figure 19 for the new coordinate systems. The mapping equations are

System I: $x = z_1 \cos 7.005^{\circ} + z_2 \sin 7.005^{\circ} - 68.3496$

$y = -z_1 \sin 7.005^{\circ} + z_2 \cos 7.005^{\circ} - 3.0155$

$z = z_3/a_3 - 2.4767$

System II: $x = z_1 \cos 11.872^{\circ} + z_2 \sin 11.872^{\circ} - 68.3968$

$y = z_1 \sin 11.872^{\circ} + z_2 \cos 11.872^{\circ} + 2.7934$

$z = z_3/a_3 + 0.4178$

with the inverse transformations given by

System I: $z_1 = (x + 68.3496) \cos 7.005^{\circ} - (y + 3.0155) \sin 7.005^{\circ}$

$z_2 = (x + 68.3496) \sin 7.005^{\circ} + (y + 3.0155) \cos 7.005^{\circ}$

$z_3 = (z + 2.4767)\, a_3$

System II: $z_1 = (x + 68.3938) \cos 11.872^{\circ} - (y - 2.7934) \sin 11.872^{\circ}$

$z_2 = (x + 68.3968) \sin 11.872^{\circ} + (y - 2.7934) \cos 11.872^{\circ}$

$z_3 = (z - 0.4178)\, a_3$

From figure 19, the surface for the system with aeration control has a very much larger minor axis than the surface for the system without aeration control while the major axis is approximately the same for both cases. All three optimum solutions of the objective functionals of figure 19 a) appear quite close to the major axis in the back quadrants, and only two optimum solutions of figure 19 b) appear in the same region with the third solution corresponding to the functional, $\phi = a_3\, z_1 \cdot z_2 + z_3$, near the minor axis. For the system without aeration control the solution

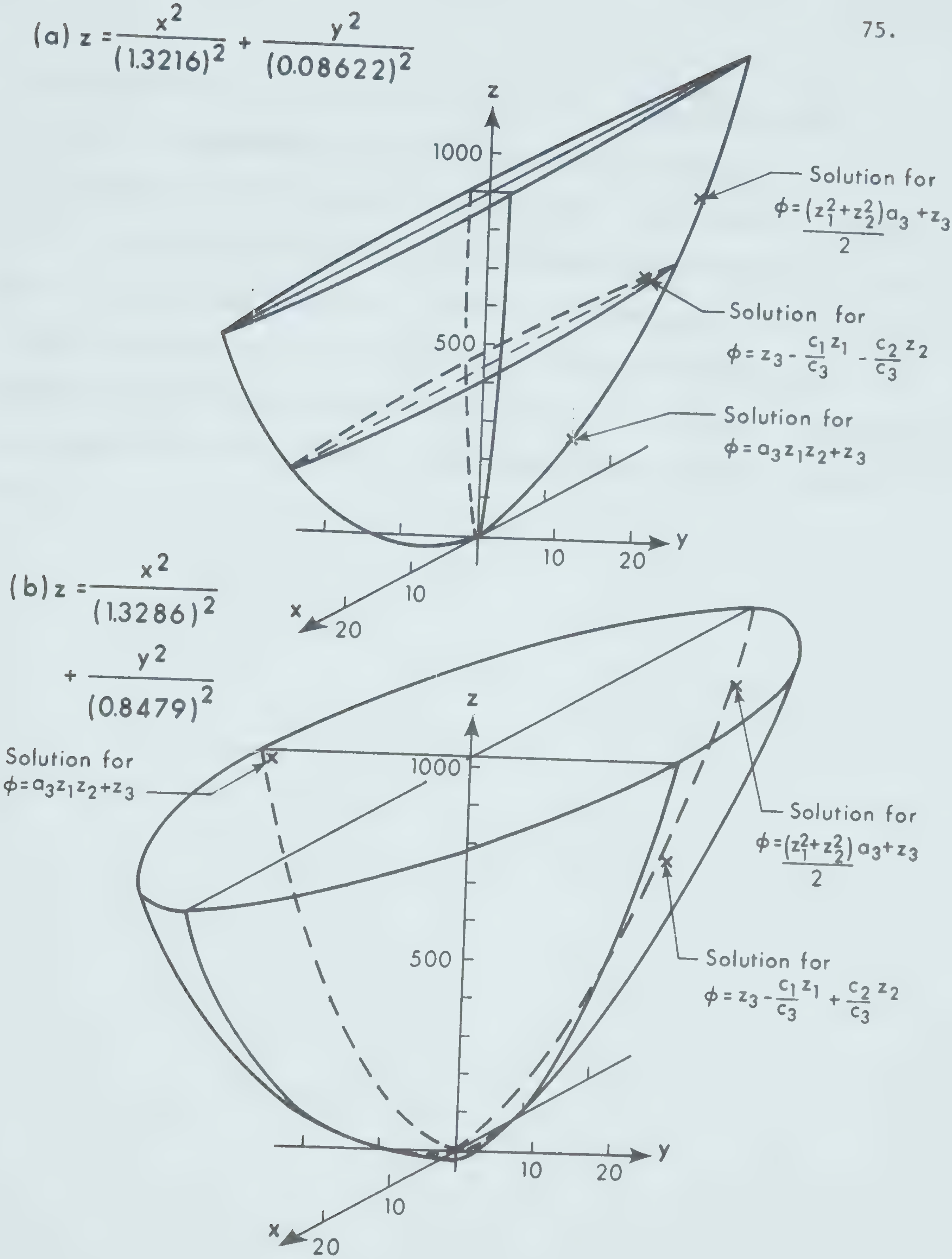

FIGURE 19: Surfaces of Performance Indices

of the functional, $\phi = a_3 z_1 \cdot z_2 + z_3$ is the closest to the local minimum of the performance index surface while the functional, $\phi = a_3 (z_1^2 + z_2^2)/2 + z_3$, is the furthest away. The order is changed for the system with the aeration control, since the optimum solution of $\phi = z_3 - (c_1/c_3) z_1 - (c_2/c_3) z_3$ is now the closest and $\phi = a_3 z_1 \cdot z_2 + z_3$ is the furthest away. The significance of these observations is to provide a means of determining the best objective functional by considering the location in relation to the local minimum of the optimum solution for ϕ.

CHAPTER (5)

CONCLUSIONS

5.1 Summary of Results

In the previous chapter, we determined the optimal water quality management policy for a section of a river using multi-cost control techniques. Two cases were considered: first, with one control, namely the dumping of effluents (BOD) only, and second, with two controls, the dumping of effluents (BOD) and the addition of artificial aeration (DO). The results obtained are summarized below:

I. System with One Control

(a) System Model

$$\begin{bmatrix} \dot{x}_1(y) \\ \dot{x}_2(y) \end{bmatrix} = \begin{bmatrix} -B & 0 \\ C & -A \end{bmatrix} \begin{bmatrix} x_1(y) \\ x_2(y) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u_1(y)]$$

where A, B and C are defined in (2-11)

(b) Performance Indices

(i) $J_1 = x_1(y_f)$

(ii) $J_2 = x_2(y_f)$

(iii) $J_3 = \int_0^{y_f} (a_1 - a_2 u_1 + a_3 u_1^2)\, dy$

(c) Objective Functionals Used

(i) $\phi = z_3 - (c_1/c_3)\, z_1 - (c_2/c_3)\, z_2$

(ii) $\phi = a_3\, z_1 \cdot z_2 + z_3$

(iii) $\phi = a_3 \cdot \dfrac{(z_1^{\,2} + z_2^{\,2})}{2} + z_3$

(d) Optimum Controls for Each Objective Functional

(i) $u_1 = 23.0 - 39.9\exp[.16(y-2.5)] + 43.4\exp[.66(y-2.5)]$

(ii) $u_1 = 23.0 - 13.3\exp[.16(y-2.5)] + 8.6\exp[.66(y-2.5)]$

(iii) $u_1 = 23.0 - 19.1\exp[.16(y-2.5)] + 1.2\exp[.66(y-2.5)]$

II. System with Two Controls

(a) System Model

$$\begin{bmatrix} \dot{x}_1(y) \\ \dot{x}_2(y) \end{bmatrix} = \begin{bmatrix} -B & 0 \\ C & -A \end{bmatrix} \begin{bmatrix} x_1(y) \\ x_2(y) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1(y) \\ u_2(y) \end{bmatrix}$$

(b) Performance Indices

(i) $J_1 = x_1(y_f)$

(ii) $J_2 = x_2(y_f)$

(iii) $J_3 = \displaystyle\int_0^{y_f} \left\{ a_1 + [-a_2 \;\; 0] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + [u_1 \;\; u_2] \begin{bmatrix} a_3 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\} dy$

(c) Objective Functionals Used

(i) $\phi = z_3 - (c_1/c_3)\, z_1 - (c_2/c_3)\, z_2$

(ii) $\phi = a_3\, z_1 \cdot z_2 + z_3$

(iii) $\phi = a_3 \dfrac{(z_1^2 + z_2^2)}{2} + z_3$

(d) Optimum Controls for Each Objective Functional

(i) $u_1 = 23.0 - 13.3 \exp[.16(y-2.5)] + 0.45 \exp[.66(y-2.5)]$

$u_2 = 1.39 \exp[.66(y-2.5)]$

(ii) $u_1 = 23.0 - 4.2 \exp[.16(y-2.5)] + 11.6 \exp[.66(y-2.5)]$

$u_2 = 36.3 \exp[.66(y-2.5)]$

(iii) $u_1 = 23.0 - 18.9 \exp[.16(y-2.5)] + 0.87 \exp[.66(y-2.5)]$

$u_2 = 2.73 \exp[.66(y-2.5)]$

5.2 Discussion of Results

A study of the results shows that the optimum dumping controls for the first two objective functionals dump more organic waste in the second half of the river section. The opposite is true for the dumping controls of the third functional. This is due to the increased weighting placed on the final conditions in the third objective functional. The aeration controls are the same for all the functionals and differ only in magnitude.

Now the dumping and aeration control problems of Chapter 4 can be compared to work done by Perlis and Cook [18]. The optimum controls which Perlis and Cook calculated according to their objective functional (see equation (3-6)) are quite different from the ones determined here

for the three objective functionals given by (4-14), (4-15) and (4-16). Their dumping control has an initial and a final value of zero with the majority of waste released in the latter half of the river section. The BOD profile decreases in magnitude along the river section which is opposite to the increasing BOD profiles determined in the previous chapter. Their aeration control has a non-zero initial value and a zero final value with the majority of oxygen released in the first half of the river section. The DO profile obtained by them are similar to the results obtained in Chapter 4. Both have the same concave upward appearance.

The optimization theory for multi-cost system in Chapter 4 has given an overall optimum control policy for the with and without aeration cases. This method of arriving at an optimum control policy for the management of the water quality has proven to be more flexible in its approach then previous papers on the subject. The results have contributed to better understanding of the behaviour of the system for different objective functionals through the investigation carried out in the previous chapter.

5.3 Suggestions for Further Research

The discussion in this thesis is not intended to set down a comprehensive Water Quality Management Policy; what has been attempted is the outlining of a procedure to be followed to obtain one such policy. We have shown some of the possible results that might be arrived at by using multi-cost control techniques. The following are some areas for further work.

(1) In Chapter 2, a bilinear model was introduced for comparison with the lumped parameter model, which showed close agreement over the limited range investigated. A rigorous study could be conducted in this area, followed by the application of multi-cost system theory to determine the necessary optimum controls.

(2) The work carried out in Chapter 4 can be extended further by the introduction of some addition objective functionals and the determination of the resulting optimum weighting factor ratios.

(3) In section 4.6 of Chapter 4, it was shown that the "n" performance indices can be related to one another by eliminating the "n-1" weighting factor ratios to give a surface. Each point of this performance index surface corresponds to a solution for an objective functional. When the surface has a local minimum, there would be an optimum functional. Using this approach, the multi-cost theory could be extended without knowing or defining optimum objective functional.

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APPENDIX I

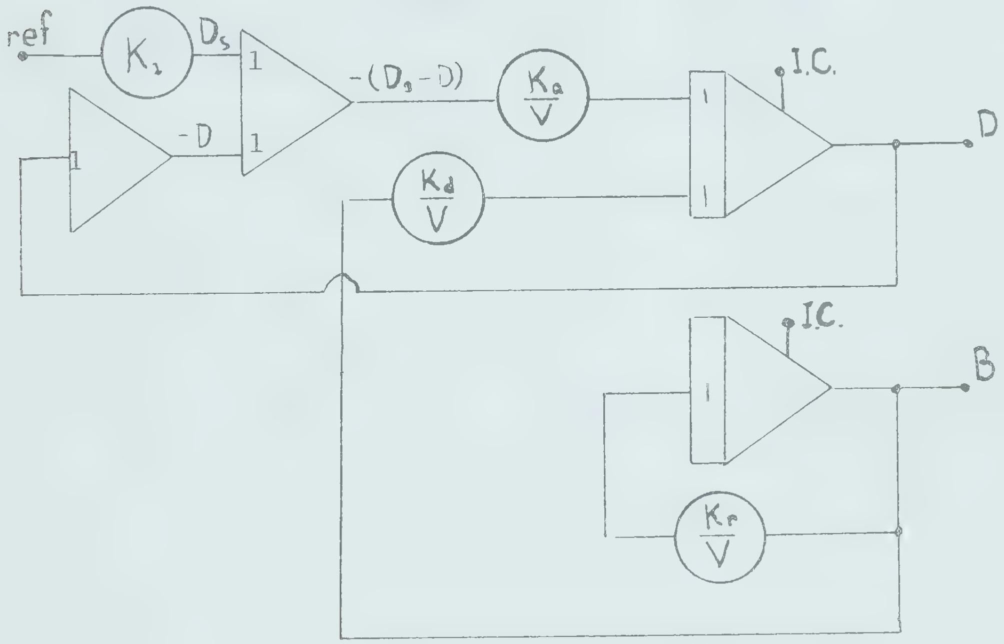
Solution of Homogeneous Equations by the Analog Computer

System 1:

$$\dot{B} = -\frac{Kr}{V} B$$

$$\dot{D} = -\frac{Kd}{V} B - \frac{Ka}{V} D + \frac{Ka}{V} D_s$$

Computer Diagram for the Solution of System 1



Parameters Used

$K_r = 0.00 \rightarrow 0.16$ 1/day $\quad D_s = 9.00$ mg/ℓ $\quad D(0) = 6.00$ mg/ℓ

$K_d = 0.16$ 1/day $\quad V = 1.00$ mg/ℓ $\quad B(0) = 15.00$ mg/ℓ

$K_a = 0.66$ 1/day

Results

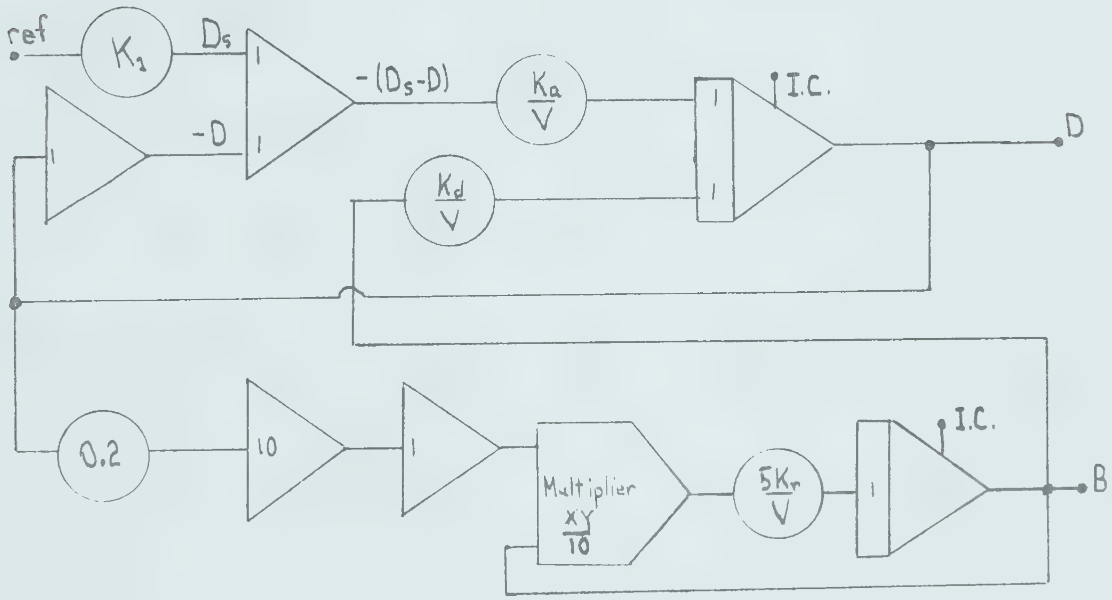
See figure (4).

System 2:

$$\dot{B} = -\frac{K}{V} D \cdot B$$

$$\dot{D} = -\frac{K_d}{V} B - \frac{K_a}{V} D + \frac{K_a}{V} D_s$$

Computer Diagram for the Solution of System 2



Parameters Used

$K = 0.00 \rightarrow 0.05$ ℓ/day·mg

$K_d = 0.16$ 1/day

$K_a = 0.66$ 1/day

$D_s = 9.00$ mg/ℓ

$V = 1.00$ mile/day

$D(0) = 6.00$ mg/ℓ

$B(0) = 15.00$ mg/ℓ

Results

See figure (5).

APPENDIX II

Comparison of Lumped Parameter and Bilinear Models

System 1:

$$\dot{B}_1 = -\frac{K_r}{V} B_1$$

$$\dot{D}_1 = -\frac{K_d}{V} B_1 - \frac{K_a}{V} D_1 + \frac{K_a}{V} D_s$$

System 2:

$$\dot{B}_2 = -\frac{K}{V} B_2 \cdot D_2$$

$$\dot{D}_2 = -\frac{K_d}{V} B_2 - \frac{K_a}{V} D_2 + \frac{K_a}{V} D_s$$

Parameters Used

K_r = variable	$K = 1.41 \cdot K_r/D_s$	D_s = 9.00 mg/ℓ	D(0) = 6.00 mg/ℓ
K_d = variable	K_a = variable	V = 1.00 mile/day	B(0) = 15.00 mg/ℓ

Solution of Homogeneous Equation by Runge-Kutta Method

Distance	$K_d = 0.16$, $K_a = 0.66$, $K_r = 0.16$				$K_d = 0.16$, $K_a = 0.66$, $K_r = 0.08$			
Miles	B_1, mg/ℓ	B_2, mg/ℓ	D_1, mg/ℓ	D_2, mg/ℓ	B_1, mg/ℓ	B_2, mg/ℓ	D_1, mg/ℓ	D_2, mg/ℓ
0.0	15.00	15.00	6.00	6.00	15.00	15.00	6.00	6.00
5.0	6.74	6.91	6.90	6.85	10.06	10.37	6.27	6.20
10.0	3.03	2.69	8.03	8.09	6.74	6.83	7.14	7.10
15.0	1.36	0.94	8.56	8.67	4.52	4.28	7.75	7.78
20.0	0.61	0.31	8.80	8.89	3.03	2.58	8.16	8.26
25.0	0.28	0.10	8.91	8.96	2.03	1.53	8.44	8.56
	$K_d = 0.16$, $K_a = 1.00$, $K_r = 0.16$				$K_d = 0.08$, $K_a = 0.16$, $K_r = 0.16$			
0.0	15.00	15.00	6.00	6.00	15.00	15.00	6.00	6.00
5.0	6.74	6.22	7.71	7.78	6.74	6.09	7.90	7.95
10.0	3.03	2.22	8.42	8.55	3.03	2.14	8.51	8.62
15.0	1.36	0.74	8.74	8.85	1.36	0.72	8.78	8.87
20.0	0.61	0.24	8.88	8.95	0.61	0.23	8.90	8.96
25.0	0.28	0.08	8.95	8.98	0.28	0.08	8.96	8.99